Market for Manipulable Information*

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Abstract

We study how market structure shapes the damage from manipulating data. A monopolist sells investors a score built on a manipulable firm signal; the firm manipulates in proportion to the price response to the score, which market clearing determines. When manipulation incentives are predictable, the seller absorbs manipulation-induced noise through score design. When they are highly uncertain, manipulation reduces score and price informativeness. In this regime, price-inelastic non-fundamental demand amplifies manipulation, more informed capital can reduce price informativeness, score-based mandates inflate scores and degrade information, and manipulation crowds out investment. The seller under-supplies verification, leaving room for enforcement.

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1 Introduction

Financial markets are in an age of information explosion. Credit ratings, ESG ratings, and other commercial scores used to forecast firm performance or certify firm characteristics increasingly rely on "alternative data," such as web traffic, app downloads, social media sentiment, or satellite imagery. Compared with traditional financial and accounting data, which are subject to established reporting and auditing requirements, these inputs tend to be more timely but are rarely verified, which makes them cheaper to manipulate. At the same time, the investors who trade on these scores have changed. Aggregate demand for stocks has become less price elastic (Koijen and Yogo, 2019; Gabaix and Koijen, 2021), a growing share of capital is managed passively or against benchmarks (Pavlova and Sikorskaya, 2023; Haddad et al., 2025), and a growing amount of capital is tied to scores directly through investment mandates and score-based indices (Morningstar, 2026).

This paper asks how these developments interact. When the data behind a score can be manipulated, how much does manipulation degrade the information in scores and prices, and how does the answer depend on the structure of the market: who trades, how price elastic their demand is, whether their demand is tied to the score itself, and how the market for information is organized? We show that market structure determines how much damage manipulation does, with implications for price efficiency, the effectiveness of characteristic-based investing, whether voluntary or mandated, and real investment.

Our model builds on Admati and Pfleiderer (1986) and adds two features: manipulable information and heterogeneous investors. The terminal payoff of a stock, $\tilde{y}$, contains a predictable component $\tilde{g}$. A monopolistic information seller collects a signal about $\tilde{g}$ from the firm, designs a score, and sells it to investors. The firm manager can privately and at a cost manipulate the signal.[1] His incentive to manipulate is also private information (Fischer and Verrecchia, 2000; Einhorn, 2007); for example, how strongly his compensation is linked to the stock price is often not public. Fundamental investors trade on their beliefs about $\tilde{y}$ and decide whether to buy the score to learn

[1] For example, social media reactions to new products are used to predict company performance. A company can influence such sentiment in various ways, from paying "influencers" to promote its products to using "bots" to post fake reviews.

more. Non-fundamental investors do not process information about $\tilde{g}$. Their demand depends on the stock price and, in an extension, on the score itself. We derive both components of non-fundamental demand from a delegated portfolio problem in which fund managers are paid partly on performance relative to a benchmark, building on Pavlova and Sikorskaya (2023).

The key equilibrium object is how strongly the stock price responds to the score. The firm manager manipulates in proportion to this price response. The price response, in turn, is determined by market clearing: it depends on the share of fundamental investors, on how price elastic the demand of non-fundamental investors is, and on whether their demand responds to the score directly. Through this channel, market structure changes how much the firm manipulates, and therefore how informative the score and the price are, even when the firm's fundamentals, the data behind the score, and the manager's manipulation cost and compensation are held fixed.

Because investors anticipate the average degree of manipulation and undo the bias it creates, what destroys information is uncertainty about the manager's incentive to manipulate. Whether this uncertainty reaches the score and the price depends on the information seller. As in Admati and Pfleiderer (1986), a monopolistic seller adds noise to keep the score from becoming too revealing through the price, which would erode buyers' willingness to pay for the score. The fact that she can add noise but cannot remove the noise that manipulation creates gives two regimes.

When manipulation uncertainty is low, the equilibrium is unconstrained: the seller treats manipulation noise as a free substitute for her own and offsets it one for one, so the fraction of score buyers, the score fee, and score and price informativeness are the same as without manipulation. We call this manipulation irrelevance. Market structure still affects informativeness in this regime, but through the seller's choice of noise rather than through manipulation. When manipulation uncertainty is high, the equilibrium is constrained: the seller adds no noise, and manipulation-induced noise feeds directly into the score and the price. Which regime applies depends on the nature of the signal and on market structure. The constrained regime is more likely when the signal is only weakly related to payoffs, as is typical of alternative data, and when non-fundamental demand is price inelastic. The share of fundamental investors matters as well, in a direction that depends on that elasticity.

Our first result, which arises in the constrained regime, is that investors who never use the score shape how informative it is. More price-elastic demand from non-fundamental investors dampens the price response to the score and weakens the manager's incentive to manipulate. In the constrained equilibrium, both the score and the stock price therefore become more informative about $\tilde{g}$, even though non-fundamental investors neither observe the score nor learn from the price. In the unconstrained equilibrium, the seller offsets the change in manipulation with her own noise, so informativeness does not depend on this elasticity, as is also true without manipulation. Elasticity still matters, however, because it determines which regime applies: by limiting manipulation, more elastic demand leaves the seller room to absorb it, making the unconstrained regime more likely. Markets with inelastic non-fundamental demand are thus doubly vulnerable: they are more likely to be constrained, and within that regime manipulation does more damage. When information can be manipulated, the rise of price-inelastic institutional demand therefore matters not only for price volatility but also for how informative prices are.

Second, more informed capital need not improve price discovery. In the constrained equilibrium, raising the share of fundamental investors has two effects. More investors trade on the score, so more of its information enters the price. But fundamental investors also replace non-fundamental investors whose elastic demand dampened the price response, so the price response rises and with it manipulation. When non-fundamental demand is sufficiently elastic, the second effect eventually dominates, and price informativeness is hump-shaped in the share of fundamental investors. In the unconstrained equilibrium, price informativeness is flat in this share under partial participation and decreasing under full participation, but for a different reason: as more of the score leaks into the price, the seller adds noise to protect its value.

This result applies directly to green investing. If $\tilde{g}$ is a firm's true greenness, fundamental and non-fundamental investors are green and traditional investors, respectively, and the score is an ESG rating, then price informativeness measures how well firms' cost of capital reflects their true greenness, the channel through which green investing is meant to influence firms. A continued rise in green investing need not improve this alignment and can weaken it, because a larger green share makes prices respond more strongly to ESG ratings and so strengthens firms' incentives to

greenwash. This contrasts with Goldstein et al. (2026), in which, in any stable equilibrium, a larger share of green investors makes the price more informative to green investors.

Third, score-based mandates can improve or degrade information, depending on the regime. When non-fundamental demand rises with the reported score, as it does for funds whose holdings or benchmarks tilt toward high-score firms, the score becomes a compliance input whose level matters regardless of what it reveals. Mandate demand therefore raises the price response to the score irrespective of its precision, strengthening the manager's incentive to manipulate and inflating scores in both regimes. In the constrained equilibrium, the seller cannot offset the additional manipulation, so stronger mandates make scores less informative. Price informativeness first rises, because the score is transmitted more strongly into prices, and then falls. The range over which it rises shrinks to zero as manipulation uncertainty grows. In the unconstrained equilibrium, stronger mandates make both the score and the price more informative. Because mandate investors respond to the score regardless of its precision, added noise does less to limit leakage into the price, so the seller cuts her noise by more than the additional manipulation adds. Inflated scores can therefore accompany better or worse information, and stronger mandates can push the market into the constrained regime but never back.

In our microfoundation, mandates arise from benchmarking: fund managers are paid partly on performance relative to a benchmark that tilts toward high-score firms. Mandate strength is then the share of performance pay tied to the benchmark times how steeply the benchmark tilts toward high scores. Adding benchmark-relative pay on top of absolute pay also makes fund managers' demand less price elastic, which by our first result further amplifies manipulation. A strong link between benchmarks and scores is thus benign when firms' manipulation incentives are predictable but should be kept modest, or paired with measures that curb manipulation, when they are highly uncertain.

Fourth, manipulation can crowd out real investment. In an extension, we allow the manager to make an unobservable investment that raises $\tilde{g}$. The manager's costs of manipulation and investment are separable, so unlike in signal-jamming models (Stein, 1989), he faces no direct trade-off between manipulating and investing. But because investment and manipulation enter the score identically,

investors discount the whole score, and the manager's reward for investing is the price response to it. In the constrained equilibrium, cheaper manipulation makes the score noisier, lowers the price response, and reduces investment. In the unconstrained equilibrium, the seller absorbs the additional noise, and investment does not depend on the cost of manipulation. Where manipulation incentives are uncertain, curbing manipulation, for example by penalizing greenwashing, therefore also restores the reward to genuine improvements such as emissions reductions.

These results build on the insight of Fischer and Verrecchia (2000) and Frankel and Kartik (2019) that uncertainty about manipulation incentives destroys information, and more so the more strongly the market responds to the report. In these models, as in the signal-jamming literature (Stein, 1989), the market's response is a Bayesian weight pinned down by the statistical properties of the report and of the manipulation, so the damage from manipulation is a property of the information environment alone. A central contribution of our paper is to determine this response, and hence the incentive to manipulate, through market clearing. How much a firm manipulates then depends on who holds its stock and how they trade, which connects the manipulation literature to the evidence on inelastic markets and benchmarked capital.

Two consequences follow. First, features of the market unrelated to information determine information quality: in the constrained regime, investors who never see the score can improve its informativeness, more informed capital can reduce price discovery when the information seller does not add any noise to the score, and mandates can degrade the information they rely on. None of these effects arises with a Bayesian weight, which does not depend on who trades, or in our model without manipulation. Second, market clearing can disable the market's defense against manipulation. A Bayesian observer puts less weight on a noisier report, limiting the payoff to manipulating it; when demand responds to the score itself, as under mandates, the price keeps responding even as the score loses information. Our model also places a profit-maximizing intermediary between the report and the market, whose noise choice absorbs manipulation in the unconstrained regime.

The information seller's incentives also limit how far the market corrects manipulation on its own. When the seller can invest in verification that raises the manager's cost of manipulation, she does not verify at all in the unconstrained regime, and in the constrained regime she never verifies enough

to restore the informativeness she targets when unconstrained. The reason is that informativeness is partly a public good: a more informative score makes the price more informative, which benefits investors who do not buy the score and, through the price response, the firm's real investment, while the information that leaks into the price erodes what buyers will pay. Outside enforcement against manipulation raises informativeness in the constrained regime. In the unconstrained regime, the seller offsets it by adding noise, so enforcement only saves the resources spent on manipulation, and raising informativeness requires constraining her score design, for example through precision standards for scores.

The two regimes also explain disagreement among rating providers in different ways. When the score is a weighted average of several manipulable signals, the weights that make it most informative about $\tilde{g}$ differ from the standard weights that minimize intrinsic noise: because weight invites manipulation, they discount signals that are precise but easy to manipulate, a form of Goodhart's law (Hennessy and Goodhart, 2023), and they depend on market structure through the price response. In the constrained regime, the seller chooses these weights, so providers that assess manipulability differently disagree, more so when manipulation uncertainty is high. In the unconstrained regime, many weight vectors are equally profitable, so providers with the same inputs can produce different scores. Neither channel relies on differences in data or skill.

These results have direct implications for policies that promote characteristic-based investing. Expanding green investing, tying mandates and benchmarks to ESG scores, and the growth of price-inelastic passive demand all make prices respond more strongly to scores and push markets toward the constrained regime. Where manipulation incentives are uncertain, these policies can undermine the information they rely on unless accompanied by enforcement against manipulation of the rated firms' inputs, which rating providers will under-supply on their own and which also restores the reward to genuine investment. The value of such enforcement is highest where manipulation incentives are hardest to predict, as for ESG attributes, alternative data, and firms approaching equity issuances or earnings announcements. Where manipulation incentives are predictable, the binding limit on information quality is instead the provider's deliberate noise, and the relevant policy is a precision standard for the score.

The model also delivers testable predictions. The effect of non-fundamental demand elasticity, or of passive and benchmarked ownership, on price informativeness should be stronger where firms' manipulation incentives are harder to predict and absent where they are predictable, which distinguishes our channel from the view that passive ownership lowers price informativeness because fewer investors acquire information. Tighter links between benchmarks and scores should be associated with score inflation everywhere, but with more informative prices where manipulation incentives are predictable and less informative prices where they are highly uncertain. Enforcement against manipulation should raise real investment and price informativeness where manipulation uncertainty is high, with little effect where it is low. Finally, rating divergence that stems from different assessments of manipulability should be larger for attributes whose manipulation incentives are more uncertain.

Related literature

Our paper is closely related to the literature on information sales. Building on the seminal work of Admati and Pfleiderer (1986), which examines a monopolist information seller designing and selling payoff-relevant information to traders, we extend their framework by incorporating the firm manager's information manipulation behavior. As a result, the information seller's input is endogenous to its pricing and design decisions, in contrast to the exogenous input in Admati and Pfleiderer (1986). Recent studies investigate related settings involving monopolistic information intermediaries, including the pricing problems of data brokers who sell data to facilitate marketing (Bergemann and Bonatti, 2015), to forecast consumer characteristics (Segura-Rodriguez, 2021), or to firms with private production costs (Yang, 2022). In these studies, the intermediary's actions do not endogenously influence the input of the information it can collect, which is the distinguishing feature of our model. These studies also consider a more flexible information design by information sellers, while we restrict the seller to only adding normal noise to the underlying signal. This is motivated by tractability and by the focus of our analysis on the interaction between noise choice and manipulation incentives, rather than on the optimal information structure per se.

With the development of alternative data, a growing literature examines how data availability affects firm behavior. Begenau et al. (2018) and Farboodi and Veldkamp (2022) study how data can

influence a firm's production and lifecycle through a data feedback mechanism. Our focus differs: we examine how data availability alters a firm manager's incentives to manipulate, which in turn reduces the informativeness of both the data and the equilibrium price.

Our work also relates to the literature on information manipulation by agents with strategic concerns, including studies on earnings manipulation (Dutta and Fan, 2014; Crocker and Slemrod, 2007; Laux and Laux, 2009; Fischer and Verrecchia, 2000). Among these, Fischer and Verrecchia (2000) is closest to ours in its Gaussian setting. We differ in two respects. First, the price response to the report, and hence the incentive to manipulate, is determined by market clearing rather than by a Bayesian weight. Second, the report is designed and sold by an intermediary, so the information is neither public nor free, and the seller's pricing and noise choices alter the manager's manipulation incentives. These studies focus primarily on incentive contracts, which we take as given. Our framework also connects to Frankel and Kartik (2019), who study a firm's ability to manipulate its signals and distort payoff-relevant information, and Bonatti and Cisternas (2020), who examine a Gaussian setting in which a strategic consumer manipulates a score used by downstream firms for price discrimination. The key difference is that in our setting the score is designed and sold by a third party information intermediary. Both Ball (2025) and Goldman et al. (2022) consider information intermediaries facing endogenous manipulation, but their intermediaries' objectives differ substantially from ours.

Our model predictions relate to the literature on ESG rating divergence and greenwashing. Studies document substantial ESG rating disagreements due to differences in measurement and methodology (Berg et al., 2022; Chatterji et al., 2016), and firms may strategically alter their behavior to attract fund flows without improving performance (Cooper et al., 2005; Kaustia and Yu, 2021; Liang et al., 2022). Our model rationalizes these observations and highlights the interactions between rating divergence and greenwashing. Chen (2023) shows that stronger ESG preferences may lead to excessive greenwashing that undermines green investment. Our mandate model generates a similar prediction: stronger mandates for high ESG-score assets can reduce the informativeness of the market about firm quality, even absent the firm's investment channel. Recent theoretical work on ESG investors and asset prices includes Pástor et al. (2021), Pedersen et al. (2021), and

Goldstein et al. (2026), among others. The closest work to ours is Goldstein et al. (2026), who also employ a noisy REE framework with heterogeneous investors. The key distinction is the information environment: their model features two exogenous information sources about financial and ESG performance, and endogenous trading leads to multiple equilibria. Our model instead focuses on endogenous signal manipulation, yielding distinct predictions. For instance, in their model, price informativeness for greenness increases with green investor participation in the stable equilibria, whereas in our model, because of endogenous manipulation, this relation is non-monotonic when manipulation uncertainty is high and traditional investors' demand is sufficiently price elastic.

Our non-fundamental and mandate demand build on the literature on delegated asset management and benchmarking (see, e.g., Stoughton, 1993; Admati and Pfleiderer, 1997; Basak and Pavlova, 2013; Kashyap et al., 2023; Pavlova and Sikorskaya, 2023). We take the benchmarking contract as given and show that it matters for information quality: benchmark-relative pay makes fund managers' demand less price elastic, and benchmarks tied to scores add demand that responds to the score itself. Both raise the price response to the score and hence the firm's incentive to manipulate, and when manipulation incentives are uncertain they can weaken the link between firms' cost of capital and the characteristic the score measures.

2 Baseline Model

2.1 Setup

The model has a single period, from time 0 to time 1. There are two assets traded in the financial market, a riskless bond with unlimited supply and a risky asset (the stock of a firm) with random supply $\tilde{s} \sim N(\bar{s}, \sigma_s^2)$. We assume that the average supply of the stock is positive, i.e., $\bar{s} > 0$. The riskless bond is the numeraire, with the risk-free rate normalized to zero.

The stock generates a random terminal payoff $\tilde{y} \sim N(\bar{y}, \sigma_y^2)$ at time 1. A predictable component of this payoff is captured by the normalized variable, $\tilde{g} \sim N(0,1)$. Let $\rho \in (0,1]$ denote the correlation between $\tilde{y}$ and $\tilde{g}$ ($\tilde{g}$ is informative about the payoff whenever $\rho \neq 0$; without loss of

generality, we restrict $\rho > 0$), then the payoff $\tilde{y}$ can be rewritten as

$$\tilde{y} = \bar{y} + \rho\sigma_y\tilde{g} + \epsilon_y, \tag{1}$$

where $\epsilon_y \sim N\left(0, \sigma_y^2(1-\rho^2)\right)$ is independent of $\tilde{g}$ and captures the unpredictable component of the payoff.

Information seller. There is a monopolistic information seller who collects a signal produced by the firm that is informative about $\tilde{g}$. Based on the signal, the information seller constructs a score, g_r, and sells it to investors at price Φ. In the absence of any manipulation by the firm, the signal about $\tilde{g}$ takes the form $\tilde{g} + \epsilon$, where $\epsilon \sim N\left(0, \underline{\sigma}^2\right)$. Here $\underline{\sigma} > 0$ represents the amount of intrinsic noise in the signal. Motivated by Admati and Pfleiderer (1986), we assume that the information seller can add independent noise to the score, and thus the score designed by the information seller can be written as $\tilde{g} + u$, where $u \sim N\left(0, \sigma_u^2\right)$ with $\sigma_u \in [\underline{\sigma}, \infty)$ combines the intrinsic noise in the signal and any potential noise added.[2] The information seller chooses the score fee Φ and noise level σ_u to maximize her profit.

Investors. There is a continuum of fundamental investors with total mass $\theta \in (0, 1]$, and a continuum of non-fundamental investors with total mass $(1-\theta)$. We require $\theta > 0$ so that there is a positive mass of potential score buyers; $\theta = 1$ is the special case without non-fundamental investors. Fundamental investors' portfolio decisions are based on their beliefs about the terminal payoff $\tilde{y}$. Each fundamental investor is endowed with initial wealth W_0 and has CARA utility. At time 0, a fundamental investor chooses holdings (ϕ, l) in the stock and a risk-free bond, respectively, to maximize their expected utility, which is given by

$$U = \mathbb{E}\left[-e^{-A(\phi\tilde{y}+l)}|\mathcal{F}\right], \tag{2}$$

[2]We rule out the possibility of the seller directly biasing the score. We interpret this restriction as a transparency or auditability constraint: the scoring algorithm is observable, so any systematic distortion would be detected and penalized. We assume the penalty is sufficiently large to make such deviations unprofitable. Berg et al. (2021) discusses incentives for information intermediaries, including rating agencies, to distort reported assessments. In our environment, investors are rational and can net out any constant bias in the score about $\tilde{g}$, regardless of whether it originates from the firm or from the information seller.

where $A > 0$ is the coefficient of absolute risk aversion, and $\mathcal{F}$ denotes the investor's information set at time 0. Among the fundamental investors, a fraction λ endogenously choose to buy the score from the information seller. We refer to them as *informed* fundamental investors. The remaining fraction $1 - \lambda$, the *uninformed* fundamental investors, do not buy the score. This causes the information set to differ between the two types of fundamental investors.

Different from fundamental investors, non-fundamental investors' demand passively responds to stock price. Each non-fundamental investor's demand for the risky asset takes the linear form

$$D(p) = D_0 - D_1 p, \tag{3}$$

where p denotes the stock price, $D_0 > 0$, and $D_1 > 0$. The intercept D_0 captures the non-fundamental investor's baseline allocation to equities. The slope D_1 drives the price elasticity of demand for the non-fundamental investor.[3]

Demand of the form in (3) arises in several settings. The first is delegated investment. In Appendix A, building on Pavlova and Sikorskaya (2023), we derive (3) from the portfolio choice of a fund manager whose compensation depends on performance, and links D_0 and D_1 to the manager's risk aversion and the terms of her contract.[4] The second is a predetermined position target. Equal-weight index funds, balanced funds, and target-date funds commit to fixed portfolio weights. They buy as the price falls and sell as it rises, without regard to what the price change reveals (Perold and Sharpe, 1988; Parker et al., 2023).[5] The third is liquidity provision. Market makers absorb order imbalances at a price concession that compensates them for inventory risk rather than reflecting any inference from the price (Grossman and Miller, 1988). Across these settings, a higher D_1 corresponds to more elastic demand, and differences in D_1 reflect differences in non-fundamental investment styles.

[3]Strictly speaking, D_1 is the slope of the demand curve rather than its price elasticity: for $D(p) = D_0 - D_1 p$, the price elasticity is $\epsilon(p) = D_1 p/(D_0 - D_1 p)$, which also depends on p and D_0. However, at any fixed (p, D_0) with positive demand, ϵ is strictly increasing in D_1, so comparative statics with respect to D_1 move in the same direction as those with respect to the price elasticity. For brevity we therefore use the two interchangeably throughout the paper.

[4]The appendix derives the demand of the mandate investors in Section 4.2, whose compensation also depends on relative performance to a score-based benchmark. Remark A.1 shows that (3) is the special case without the relative performance component.

[5]Equal-weight index funds exhibit a higher effective D_1 than cap-weighted funds because any price change pushes a stock away from its target weight, triggering mechanical rebalancing.

Non-fundamental investors' holdings in the riskless bond are irrelevant in our model.

Firm. The key novel feature of our model is the possibility for the firm's manager to manipulate the signal about $\tilde{g}$. Specifically, the firm manager can privately change the level of the signal by δ at a cost $\frac{c}{2}\delta^2$, where $c > 0$ is the cost parameter for signal manipulation. The cost represents the manager's effort to manipulate, or potential legal and reputational penalties from doing so. The firm manager's optimization problem is

$$\max_{\delta} \; q\mathbb{E}\left(p\right) - \frac{c}{2}\delta^2, \tag{4}$$

where $q \sim N\left(\bar{q}, \sigma_q^2\right)$ represents the manager's incentive to inflate the stock price which is independent of all other random variables in the model. For instance, q could capture the sensitivity of the manager's compensation to the stock price, which may arise from an optimal contract designed to address other independent agency problems. Alternatively, q could reflect implicit incentives such as career concerns (Holmström, 1999).[6] The realization of q is privately observed by the firm manager. We assume that $\sigma_q > 0$ and $\bar{q} > 0$, so that the firm manager on average tends to inflate the stock price. Notice that q being normally distributed means that it could be negative, in which case the manager wants to reduce the stock price. An example is when the management team is considering a management buyout (Perry and Williams, 1994).[7] In the manager's problem Eq. (4), it is clear that only $\frac{q}{c}$ matters in his incentive, with both stronger incentive-based compensation and lower cost of manipulation leading to more aggressive manipulation.

After signal manipulation, the score generated by the information seller, g_r, becomes

$$g_r = \tilde{g} + u + \delta. \tag{5}$$

While investors and the information seller are aware of the firm manager's incentives to manipulate the signals, they cannot precisely forecast the amount of manipulation by the firm manager and

[6] For example, the market may infer the manager's ability from the stock price, and q can capture both how much the manager values outside options and the market's assessment of his ability.

[7] In this formulation, we assume q to be random and c to be deterministic. However, introducing uncertainty in the cost of manipulation c would have qualitatively similar effects as uncertainty in q, the incentive to increase stock prices.

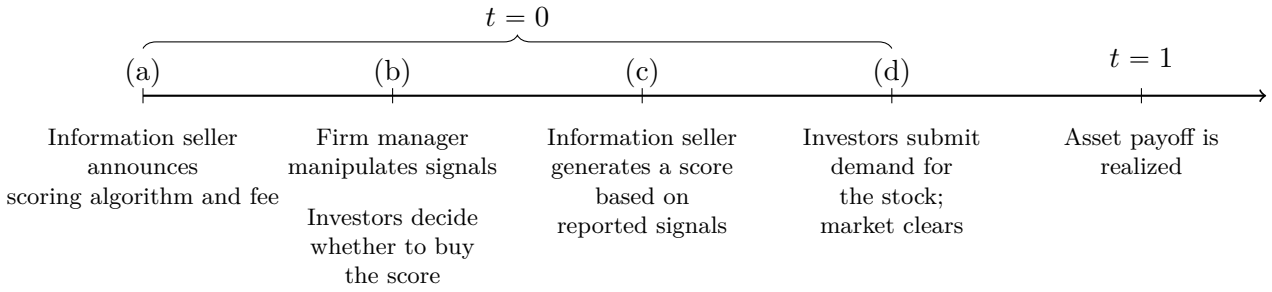


Figure 1: Model Timeline

remove its effect from the score entirely because the realization of q is unobservable to them.

Timing. Figure 1 summarizes the timeline of the model: At time 0, the information seller chooses both the noise σ_u and the score fee Φ. Then the firm manager decides on the level of manipulation δ, and fundamental investors decide whether to buy the score from the information seller. After that, all investors submit their demand functions and the price p is endogenously determined in equilibrium. At time 1, the asset payoff is realized, and all players collect their payoffs.

Next, we define the market equilibrium in the presence of information manipulation.

Definition 1. *We define an equilibrium as* $\{\Phi^*, \sigma_u^*, \lambda^*, p^*, \delta^*\}$*, where* Φ^* *is the score fee chosen by the information seller,* σ_u^* *is the standard deviation of the noise in the score chosen by the information seller,* λ^* *is the fraction of fundamental investors who opt to purchase the score, and* δ^* *is the manipulation level that the firm manager selects, such that the following conditions hold:*

1. *a fraction* λ^* *of the fundamental investors purchase the score, while the remaining fraction* $(1-\lambda^*)$ *choose not to purchase the score, to maximize their utility Eq.* (2)*;*

2. *the firm manager selects* δ^* *to maximize the objective Eq.* (4)*;*

3. *the information seller sets the score fee* Φ^* *and the standard deviation of the noise in the score* σ_u^* *to maximize her profit from score sales;*

4. *the equilibrium price* p^* *clears the market, i.e., the total demand for the stock by all investors equals supply* $\tilde{s}$*.*

2.2 Equilibrium Characterization

We solve for the market equilibrium with manipulation in two steps. First, we characterize the partial equilibrium taking the information seller's strategy as given, namely the noise level σ_u and the fraction λ of fundamental investors who buy the score. In this step, the score fee Φ is pinned down endogenously by investors' participation decisions. Second, we solve the information seller's problem and endogenize (σ_u, λ), or equivalently (σ_u, Φ).

We restrict attention to equilibria in which the asset price is linear in the relevant state variables. The following proposition summarizes the partial equilibrium characterization.

Proposition 1. *Suppose that the information seller chooses noise level σ_u and sets the price for the score such that a fraction $\lambda \in (0, 1]$ of fundamental investors choose to buy the score. Then there exists an equilibrium such that:*

1. *the firm manager's manipulation strategy is*

$$\delta = a_r \frac{q}{c}, \tag{6}$$

2. *the equilibrium score is*

$$g_r = \tilde{g} + u + a_r \frac{q}{c}, \tag{7}$$

3. *the equilibrium price is*

$$p = a_0 + a_r g_r + a_s \tilde{s}, \tag{8}$$

where a_0, $a_r > 0$ and $a_s < 0$ are uniquely solved by (A.28), (A.30) and (A.32) in the appendix.

The key variable that affects the firm manager's manipulation incentive and price informativeness is a_r. First, as Eq. (8) shows, an increase in a_r tends to increase the correlation between the stock price p and the score g_r (assuming a_s is unchanged), which tends to make the price more informative

(at least in the absence of information manipulation). Second, an increase in a_r also intensifies the firm manager's manipulation (see Eq. (6)), which tends to reduce the informativeness of the score, and in turn the informativeness of the stock price. The trade-off between these two opposing effects is key to our analysis.

Note that, $\bar{q}$, which represents the manager's average incentive to inflate the stock price, has no effect on either the equilibrium price or the informativeness of the score or the price. While a higher $\bar{q}$ does raise the average score level, its effect on the equilibrium price is exactly offset by a_0. Intuitively, fundamental investors care about $\tilde{g}$. When they update beliefs about $\tilde{g}$ using the score, they account for the perceived manipulation activity. Because the average inflation incentive is fully anticipated, it does not affect learning. As a result, the manipulation incentive q affects the equilibrium price and the informativeness of both the score and the price only through its uncertainty σ_q. Moreover, a higher σ_q reduces the informativeness of both the score and the price, as shown in the following result.

Corollary 1. *Fix the information seller's choices σ_u and λ. As σ_q increases, a_r decreases,* $\mathrm{Var}(\delta^*) = a_r^2 \frac{\sigma_q^2}{c^2}$ *increases, and each of* $\mathrm{corr}(g_r, \tilde{g})$, $\mathrm{corr}(p, \tilde{g})$, *and* $\mathrm{corr}(p, g_r)$ *decreases.*

Corollary 1 is a partial equilibrium result. It confirms that σ_q is the key source in the manager's manipulation incentive q that drives the informativeness in equilibrium. It shows that a higher σ_q reduces the sensitivity of price to the score, makes both the score g_r and the price p less informative about $\tilde{g}$, and also lowers the correlation between p and g_r.

To understand these comparative statics, consider how an increase in σ_q changes the noise in the score. A direct effect of a higher σ_q is to make the score noisier, because the noise introduced by signal manipulation is proportional to σ_q, holding a_r fixed. An indirect effect is that a higher σ_q makes the price less responsive to the score, that is, a_r decreases. This, in turn, weakens the firm manager's incentive to manipulate the signal, so $|\delta^*|$ decreases holding q fixed. However, because the direct effect dominates, the overall uncertainty in manipulation, captured by $\mathrm{Var}(\delta^*) = a_r^2 \frac{\sigma_q^2}{c^2}$, still increases. This is why the score becomes a noisier predictor of the payoff component $\tilde{g}$. Finally, since the information in price about $\tilde{g}$ ultimately stems from the score, it follows that the correlation between price and $\tilde{g}$ also decreases with σ_q.

Next, we turn to the information seller's optimization problem. First, we examine how the score fee is determined. This is a classical result from Admati and Pfleiderer (1986).

Lemma 1 (Admati and Pfleiderer (1986))**.** *Let $\mathscr{F}_U$ and $\mathscr{F}_I$ be the time 0 information sets of uninformed and informed fundamental investors, respectively. For any given λ, the equilibrium score fee is*

$$\Phi(\lambda) = \frac{1}{2A} \ln\left(\frac{\mathrm{Var}(\tilde{y} \mid \mathscr{F}_U)}{\mathrm{Var}(\tilde{y} \mid \mathscr{F}_I)}\right). \tag{9}$$

The information seller's payoff is $\theta\lambda\Phi(\lambda)$.

Notice that the dependence of the score fee Φ on λ in Eq. (9) is implicit. The difference between the two information sets $\mathscr{F}_U$ and $\mathscr{F}_I$ is the score. As the fraction of investors who choose to purchase the score changes, so does the informativeness of the stock price, which both types of investors observe, relative to that of the score. This relative informativeness determines the right-hand-side of Eq. (9).

Since determining the score fee is equivalent to determining the equilibrium fraction of score buyers, we can rewrite the information seller's problem in the following equivalent form:

$$\max_{\lambda\in[0,1],\, \sigma_u\in[\underline{\sigma},\infty)} \lambda \cdot \frac{1}{2A} \ln\left(\frac{\mathrm{Var}(\tilde{y} \mid \mathscr{F}_U)}{\mathrm{Var}(\tilde{y} \mid \mathscr{F}_I)}\right). \tag{10}$$

Before presenting the full equilibrium solution, we introduce a new variable $M(\lambda)$, which is defined as the solution of

$$\rho^2 + \frac{\rho\,\theta\lambda}{A\sigma_y} M(\lambda) = 1 + \underline{\sigma}^2 + \left(\frac{1}{\frac{\rho}{\sigma_y} + \left(\frac{\theta}{A\sigma_y^2} + (1-\theta)D_1\right) M(\lambda) \cdot \frac{A\sigma_s^2\sigma_y^2 M(\lambda) + \rho\sigma_y\theta\lambda}{A\sigma_s^2\sigma_y^2 M(\lambda) + \rho\sigma_y\theta}}\right)^2 \frac{\sigma_q^2}{c^2}. \tag{11}$$

Note that the right hand side of Eq. (11) equals $\mathrm{Var}(g_r)$ when the information seller chooses $\sigma_u = \underline{\sigma}$. The interpretation of $M(\lambda)$ is as follows. The equilibrium stock price can be rewritten as

$$p = a_0 + a_r\left(g_r + \frac{a_s}{a_r}\tilde{s}\right).$$

The ratio $-\frac{a_s}{a_r} > 0$ governs how informative the price is about the score: a lower value of $-\frac{a_s}{a_r}$ means that the price loads more heavily on the score relative to the supply noise, making the price more informative about g_r. Fixing λ, the information seller can improve the price informativeness about the score by reducing σ_u, which lowers the equilibrium ratio $-\frac{a_s}{a_r}$. The quantity $M(\lambda)$ is the lowest value of $-\frac{a_s}{a_r}$ that the seller can achieve, attained when she sets $\sigma_u = \underline{\sigma}$. In other words, $M(\lambda)$ represents the maximum price informativeness about the score that the seller can deliver for a given λ. As we show below, this object plays a key role in determining when the manager's manipulation affects equilibrium outcomes.

We now characterize the information seller's optimal choice of score fee Φ^* and noise level σ_u^*. The equilibrium takes one of three forms, which fall into two regimes: in the *unconstrained* equilibrium, with partial or full participation, the seller's noise choice is interior, $\sigma_u^* > \underline{\sigma}$; in the *constrained* equilibrium, she is at the noise floor, $\sigma_u^* = \underline{\sigma}$.

Theorem 1. *Define*

$$\overline{\lambda} \equiv \frac{\lambda_0 A \sigma_s \sigma_y}{\theta}, \tag{12}$$

where λ_0 is a constant number solved in the appendix and is independent of all other model parameters. Then the information seller's optimal choices Φ^* (and thus λ^*) and σ_u^* are characterized by three distinct cases as follows:

- ***Unconstrained equilibrium with partial participation:*** $\lambda^* < 1$.

  If $\overline{\lambda} < 1$ and $M(\overline{\lambda}) < M^* = \frac{\rho}{\sigma_s}$, then the information seller chooses σ_u^* satisfying

$$\mathrm{Var}(g_r) = 1 + \sigma_u^{*2} + \left(\frac{a_r^* \sigma_q}{c}\right)^2 = \rho^2 \left(1 + \frac{\theta\,\overline{\lambda}}{A \sigma_s \sigma_y}\right), \tag{13}$$

  and sets Φ^* so that $\lambda^* = \overline{\lambda}$. Here a_r^* is the equilibrium a_r in the price equation, given by Eq. (A.38) in the appendix.

- ***Unconstrained equilibrium with full participation:*** $\lambda^* = 1$.

If $\overline{\lambda} \geqslant 1$ *and* $M(1) < M^* = \frac{\rho}{\sigma_s}$*, then the information seller chooses* σ_u^* *satisfying*

$$\mathrm{Var}(g_r) = 1 + \sigma_u^{*2} + \left(\frac{a_r^{**}\sigma_q}{c}\right)^2 = \rho^2\left(1 + \frac{\theta}{A\sigma_s\sigma_y}\right), \tag{14}$$

and sets Φ^* *as the highest fee under which all fundamental investors buy the score, i.e.,* $\lambda^* = 1$*. Here* a_r^{**} *is the equilibrium* a_r *in the price equation, given by Eq. (A.39) in the appendix.*

- ***Constrained equilibrium.***

Otherwise, in equilibrium the information seller chooses $\sigma_u^* = \underline{\sigma}$*, and sets* Φ^* *so that a fraction* λ^* *of fundamental investors buy the score, where* λ^* *solves*

$$\lambda^* = \arg\max_{\lambda\in[0,1]} \lambda \ln\left(1 + \frac{A\sigma_s^2\sigma_y^2}{\theta\lambda} \cdot \frac{1}{\left(\frac{\rho}{M(\lambda)}\sigma_y + \frac{\theta\lambda}{A} + \frac{\sigma_s^2\sigma_y}{\rho}M(\lambda)\right)}\right), \tag{15}$$

and $-\frac{a_s}{a_r} = M(\lambda^*)$*.*

Theorem 1 shows that, depending on the model parameters, there are three possible cases. In the two unconstrained cases, the information seller chooses an interior noise level to ensure that the variance of the score g_r matches a target (Eq. (13) under partial participation and Eq. (14) under full participation). The difference is that, under partial participation, uninformed and informed fundamental investors coexist, whereas under full participation, the score is sold to all fundamental investors and thus all fundamental investors are informed. In the constrained equilibrium, the information seller does not add any noise to the score (σ_u is constrained to $\underline{\sigma}$), so the total noise level is only determined by the intrinsic noise in the signal and the noise from the firm manager's manipulation.

The two unconstrained cases differ from the constrained case in whether the seller can offset manipulation-induced noise. Proposition 2 shows that, in the unconstrained equilibrium, the fraction of score buyers, score fee, score variance, and score and price informativeness coincide with their counterparts in the benchmark without signal manipulation ($q \equiv 0$). The manager's manipulation, the seller's noise choice, and the mean score can nevertheless differ from that benchmark.

Proposition 2 (Manipulation Irrelevance)**.** *In the unconstrained equilibrium of Theorem 1, with*

either partial or full participation, the equilibrium fraction of fundamental investors who buy the score, λ^*, the score fee Φ^*, the variance of the score $\mathrm{Var}(g_r)$*, and the informativeness of both the score g_r and the price p with respect to $\tilde{g}$ are identical to those in the benchmark model without manipulation ($q \equiv 0$).*

Here, "manipulation irrelevance" refers to the fact that manipulation does not affect equilibrium price informativeness, and that stock prices and investors' asset holdings have the same distributions as in the case without information manipulation. Notice that the firm does alter its signal in these two cases, and one that chooses a larger δ (e.g., due to a stronger-than-average incentive $q > \bar{q}$) will see its stock price inflated in equilibrium.

The key mechanism behind Proposition 2 is that the information seller retains slack in her noise choice. In both unconstrained cases, the optimal noise satisfies $\sigma_u^* > \underline{\sigma}$, so the seller can absorb any manipulation induced noise by reducing her own added noise, holding the total variance of the score exactly at its revenue maximizing target. In the constrained equilibrium, this absorption channel is shut down: there is so much added noise due to the manager's manipulation that the information seller does not want to add any additional noise; instead, she sets $\sigma_u = \underline{\sigma}$. As a result, firm manager's manipulation feeds directly into the score and the price.

The following proposition formalizes the role of σ_q in determining which of the two regimes prevails.

Proposition 3. *There exist thresholds $0 \leqslant \sigma_q^\dagger \leqslant \sigma_q^{\text{full}}$, with σ_q^{full} independent of θ, such that:*

1. *The equilibrium is constrained if and only if $\sigma_q \geqslant \sigma_q^\dagger$. The closed form of $\sigma_q^\dagger$ is given in Appendix Eq. (A.51). When $\sigma_q \geqslant \sigma_q^{\text{full}}$, the equilibrium has full participation ($\lambda^* = 1$).*

2. $\mathrm{corr}(g_r, \tilde{g})$ *and* $\mathrm{corr}(p, \tilde{g})$ *are strictly decreasing in σ_q in constrained equilibrium.*

Proposition 3 distinguishes three ranges of σ_q, the uncertainty in the firm manager's manipulation incentive. When $\sigma_q < \sigma_q^\dagger$, the equilibrium is unconstrained: the information seller's noise choice has enough slack to fully absorb the manipulation-induced variance $a_r^2 \frac{\sigma_q^2}{c^2}$, so the fraction of score buyers, score fee, score variance, and score and price informativeness coincide with their counterparts in

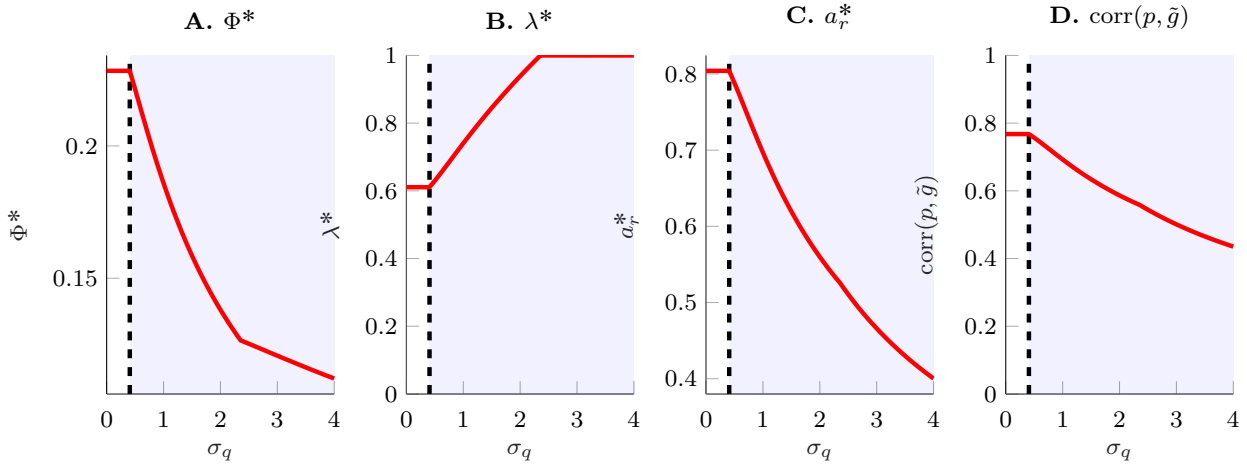


Figure 2: **Comparative statics in uncertainty of manipulation incentive σ_q.** The figure plots equilibrium outcomes as a function of σ_q. Panel A reports the equilibrium score fee Φ^*; Panel B reports the fraction of informed fundamental investors λ^*; Panel C reports the sensitivity of the equilibrium price to the score, a_r^*; Panel D reports price informativeness $\text{corr}(p, \tilde{g})$. The white region corresponds to the unconstrained equilibrium with partial participation ($\lambda^* < 1$) of Theorem 1, and the blue region corresponds to the constrained equilibrium. Vertical dashed lines mark regime boundaries. Parameter values are $\sigma_g = 1$, $\sigma_y = 1.5$, $\rho = 0.8$, $A = 1$, $\sigma_s = 0.5$, $\underline{\sigma} = 0.1$, $c = 1.5$, $\theta = 0.8$, $D_0 = 1$, and $D_1 = 1$.

the benchmark in which $q \equiv 0$. As explained following Proposition 1, the mean incentive $\bar{q}$ changes the average score but is anticipated by investors; the stock-price distribution and score and price informativeness depend on the uncertainty σ_q, not on $\bar{q}$. As σ_q rises past $\sigma_q^\dagger$, the constraint $\sigma_u \geqslant \underline{\sigma}$ binds and the economy enters the constrained regime. In the constrained equilibrium, further increases in σ_q then reduce the informativeness of both the score and the price. Once σ_q is large enough ($\sigma_q \geqslant \sigma_q^{\text{full}}$), the score is so noisy that the seller optimally sells it to all fundamental investors ($\lambda^* = 1$).

The seller's response to increasing σ_q is qualitatively similar to Admati and Pfleiderer (1986). Manipulation noise serves as a free substitute for the seller's own added noise. Only when σ_u reaches $\underline{\sigma}$ and no further absorption is possible can manipulation affect the score fee and informativeness: additional manipulation noise then pushes $\text{Var}(g_r)$ above its optimal level, the seller's profit falls, and she responds by adjusting the score fee.

As a numerical exercise, Figure 2 plots how the fee of the equilibrium score Φ^*, the fraction of informed fundamental investors λ^*, the equilibrium price sensitivity to score a_r^*, and the price

informativeness $\text{corr}(p, \tilde{g})$ change with the uncertainty of firm manager's signal manipulation incentive σ_q. At lower levels of σ_q, the equilibrium is unconstrained, and these four plotted quantities are independent of σ_q. When σ_q continues to increase, the equilibrium becomes the constrained equilibrium. An increase in σ_q makes the score more noisy, which makes the score less valuable to the score buyers, and also reduces the information spillover externality. The numerical results suggest that, to maximize profit, the information seller chooses a lower score fee. It is also intuitive that the price informativeness is lower due to higher uncertainty in firm manager's signal manipulation incentive.

Having established that the uncertainty in the firm manager's manipulation incentive, σ_q, governs whether the equilibrium is unconstrained or constrained, we now examine how the remaining model primitives shape this boundary.

2.3 When Is the Equilibrium More Likely to Be Constrained?

As Proposition 2 shows, the unconstrained equilibrium is "manipulation-irrelevant": price informativeness is the same as in the benchmark without manipulation. By Proposition 3, the equilibrium is unconstrained if and only if $\sigma_q < \sigma_q^\dagger$, so manipulation affects price informativeness only when $\sigma_q \geqslant \sigma_q^\dagger$. The threshold $\sigma_q^\dagger$ depends on the other model primitives. A lower $\sigma_q^\dagger$ means that the equilibrium is constrained for a wider range of σ_q; in this sense, we say that the equilibrium is "more likely" to be constrained when $\sigma_q^\dagger$ is smaller.

To see when signal manipulation is more likely to affect price informativeness, we examine four primitives in turn: the correlation between the predictable component and the terminal payoff (ρ), the demand elasticity of each non-fundamental investor (D_1), risk aversion (A), and the mass of fundamental investors (θ). A notable feature is that non-fundamental investors play a distinctive role: both their demand elasticity D_1 and their market share $1-\theta$ interact with manipulation in ways that have no counterpart in the benchmark without manipulation.

We begin with ρ, which measures the informativeness of the raw signal (in the absence of manipulation).

Lemma 2. *There exists $\rho^\dagger > 0$ such that the equilibrium is constrained if and only if $\rho \leqslant \rho^\dagger$.*

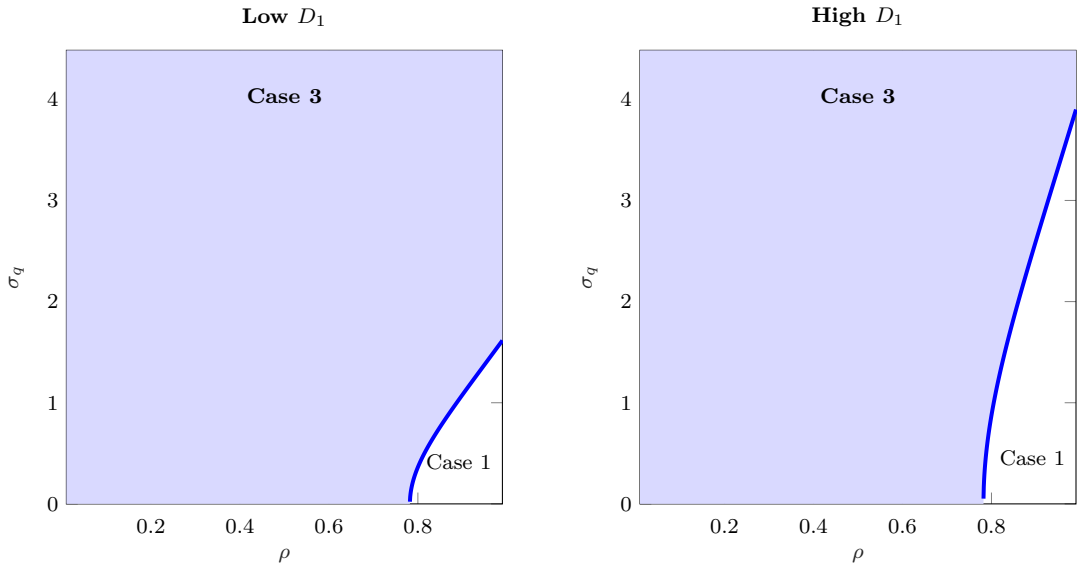


Figure 3: **Equilibrium cases in signal strength ρ.** This figure plots the regime boundary separating the unconstrained region (below the boundary) from the constrained region (above the boundary) as a function of the correlation ρ between $\tilde{g}$ and $\tilde{y}$ for two values of the non-fundamental demand elasticity D_1. The left panel sets $D_1 = 0.5$ (low D_1) and the right panel sets $D_1 = 6.67$ (high D_1). Within each panel, the blue curve is the regime boundary: below the curve (labeled Case 1) the equilibrium is the unconstrained equilibrium with partial participation of Theorem 1, and above the curve (labeled Case 3) it is the constrained equilibrium. Parameter values are $\sigma_s = 0.5$, $\sigma_y = 1.5$, $A = 1$, $c = 1.5$, $\theta = 0.8$, and $\underline{\sigma} = 0.1$.

Figure 3 illustrates Lemma 2. The mechanism operates through the information seller's incentive to add noise to the score. When ρ is high, the predictable component resolves a large share of the uncertainty in $\tilde{y}$, making the score highly valuable. Score buyers therefore trade aggressively on the score, and their trades reveal substantial information through the equilibrium price. This information spillover erodes the advantage of purchasing the score, reducing the seller's revenue. To mitigate the spillover, the seller adds considerable noise to the score, which leaves ample room to absorb any additional noise from the firm manager's manipulation. When ρ is low, $\tilde{g}$ accounts for less of the variation in $\tilde{y}$, so the score is less valuable, trading on it is less aggressive, and the resulting spillover is smaller. The seller therefore has less incentive to add noise, and her noise choice σ_u is closer to the lower bound $\underline{\sigma}$. In this region, even moderate manipulation uncertainty σ_q can exhaust the seller's remaining capacity to offset the manipulation induced variance, pushing σ_u to $\underline{\sigma}$ and triggering the constrained regime.

This result speaks to the growing use of alternative data in financial markets. Investors and data vendors increasingly rely on signals such as satellite imagery, app downloads, social media sentiment, and web traffic to forecast firm performance.[8] Each of these signals is informative but individually far less correlated with the firm's terminal cash flow than traditional financial data such as earnings or balance sheet items. In the language of our model, alternative data corresponds to a low value of ρ. Lemma 2 then implies that the firm manager's incentive to manipulate is more likely to affect score and price informativeness in this setting. The result suggests that, as the finance industry shifts toward alternative data, concerns about data integrity and signal manipulation become increasingly relevant for market equilibrium.

The next result examines how the demand elasticity of non-fundamental investors affects whether the equilibrium is constrained.

Lemma 3. *There exists* $D_1^\dagger \in [0, \infty]$ *such that the equilibrium is constrained if and only if* $D_1 \leqslant D_1^\dagger$.

Lemma 3 establishes that the equilibrium is constrained if and only if the demand elasticity of non-fundamental investors, D_1, falls below a threshold $D_1^\dagger$. Notably, this result has no analogue in the benchmark without manipulation: when $q \equiv 0$, the parameter D_1 does not affect the equilibrium structure at all. The role of D_1 emerges entirely through its interaction with the firm manager's manipulation incentive.

The intuition is as follows. Non-fundamental investors trade mechanically in response to price movements without processing information about $\tilde{g}$. When D_1 is high, each non-fundamental investor's demand is highly elastic, so any price increase triggered by an inflated score is met with a large non-fundamental sell response. This dampens the equilibrium sensitivity of the price to the score, a_r, which in turn reduces the firm manager's optimal manipulation $\delta^* = a_r \frac{q}{c}$. With less manipulation-induced noise in the score, the information seller retains sufficient slack in her noise choice, and the equilibrium is more likely to be unconstrained. The opposite is true when D_1 is low, as price becomes more sensitive to the score and the firm manager manipulates more aggressively.

[8] Grand View Research, "Alternative Data Market Size, Share & Trends Analysis Report," 2024. Available at https://www.grandviewresearch.com/industry-analysis/alternative-data-market. The report estimates global spending on alternative data at $11.65 billion in 2024, with a projected compound annual growth rate exceeding 60%, driven primarily by hedge funds and institutional investors seeking nontraditional signals such as satellite imagery, web traffic, and consumer transaction data.

The seller's capacity to absorb the resulting noise is then more easily exhausted, and the equilibrium shifts to the constrained regime.

This result highlights a new channel through which the demand elasticity of non-fundamental investors shapes asset prices by influencing the equilibrium incentives of information manipulation. Markets in which non-fundamental investors respond inelastically to price movements are more vulnerable to information manipulation.

Next, we examine the impact of fundamental investor's risk aversion A on equilibrium regime.

Lemma 4. *Let* $A_0 = \frac{\theta}{\lambda_0 \sigma_s \sigma_y}$.

1. *If* $(1-\theta)D_1\sigma_y > \sigma_s$, *then there exists* $0 \leqslant \underline{A}^\dagger \leqslant A_0 \leqslant \bar{A}^\dagger$ *such that the equilibrium is constrained if and only if* $A \in \left(0, \underline{A}^\dagger\right] \cup \left[\bar{A}^\dagger, \infty\right)$, *and is unconstrained if and only if* $A \in (\underline{A}^\dagger, \bar{A}^\dagger)$.

2. *If* $(1-\theta)D_1\sigma_y \leqslant \sigma_s$, *then there exists* $A^\dagger \in [0, \infty)$ *such that the equilibrium is constrained if and only if* $A \geqslant A^\dagger$, *and is unconstrained if and only if* $A \in (0, A^\dagger)$.

Figure 4 illustrates Lemma 4. The relationship between risk aversion and the equilibrium regime is qualitatively different from the comparative statics for ρ and D_1. The direction of the effect depends on the demand elasticity of non-fundamental investors, and when D_1 is sufficiently high, the relationship is non monotonic: the equilibrium is unconstrained only for intermediate levels of risk aversion.

We discuss the effect of risk aversion on the equilibrium regime by considering the cases $A \geqslant A_0$ and $A < A_0$ separately. Suppose the equilibrium is initially unconstrained.

When $A \geqslant A_0$, the information seller sells the score to all fundamental investors ($\lambda^* = 1$). The main force is the standard risk bearing effect. As A rises, informed fundamental investors trade less aggressively on the score. The price then becomes less informative about the score, which weakens the seller's incentive to add noise. With less noise added, the seller's slack for absorbing manipulation shrinks, and the equilibrium moves toward the constrained regime.

When $A < A_0$, the seller sells the score to a fraction $\bar{\lambda} < 1$ of fundamental investors. In this region, an increase in A leads the seller to expand the set of score buyers ($\bar{\lambda}$ rises with A). The higher informed mass offsets the weaker trading intensity of each individual informed investor. As

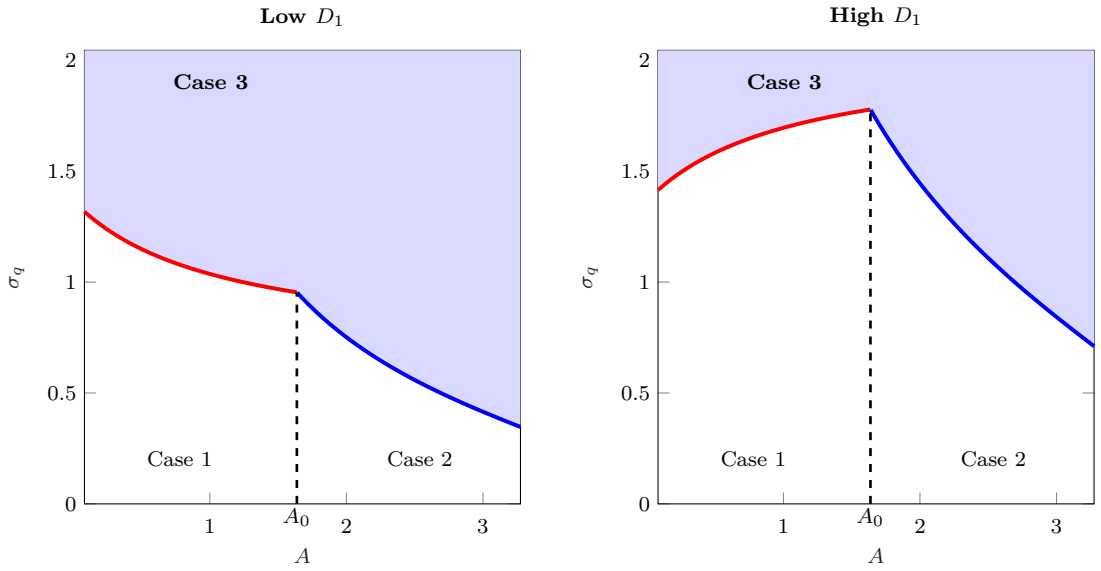


Figure 4: **Equilibrium cases in risk aversion A.** This figure plots the regime boundary separating the unconstrained region (below the boundary) from the constrained region (above the boundary) as a function of risk aversion A for two values of the non-fundamental demand elasticity D_1. The left panel sets $D_1 = 0.33$, so that $(1-\theta)D_1\sigma_y < \sigma_s$, and the right panel sets $D_1 = 3$, so that $(1-\theta)D_1\sigma_y > \sigma_s$. Within each panel, the red curve is the boundary segment corresponding to the region $A \leqslant A_0$ (where $\bar{\lambda} \leqslant 1$), the blue curve is the boundary segment corresponding to $A > A_0$ (where $\bar{\lambda} > 1$), and the vertical dashed line marks $A = A_0$. Labels inside each panel indicate the equilibrium regime of Theorem 1 in each region: Case 1 is the unconstrained equilibrium with partial participation, Case 2 the unconstrained equilibrium with full participation, and Case 3 the constrained equilibrium. Parameter values are $\rho = 0.9$, $c = 1.5$, $\sigma_s = 0.5$, $\sigma_y = 1.5$, $\theta = 0.8$, and $\underline{\sigma} = 0.1$.

in Admati and Pfleiderer (1986), these two effects cancel, so in the absence of manipulation the informativeness of the price about the score is unaffected by A.

With manipulation, however, two additional forces arise. The first is an aggregate market clearing effect. As A rises, all fundamental investors become less elastic to price changes, so a given change in the score must induce a larger price response to clear the market. A stronger price response strengthens the firm manager's manipulation incentive and therefore pushes the equilibrium toward the constrained region. The second force works through uninformed fundamental investors, who learn from the score information embedded in the price. When the price is more informative about the score, uninformed investors trade more aggressively on their inference, which further amplifies the price response to the score and strengthens the firm manager's manipulation incentive. A higher risk aversion A dampens this response by making uninformed investors less willing to trade

on the information revealed by the price. Through this channel, a higher A weakens manipulation and pushes the equilibrium toward the unconstrained region.

Which of these two forces dominates depends on D_1. When non-fundamental demand is highly price elastic (high D_1), market clearing relies less on the trading intensity of fundamental investors, so the first channel is weaker. In that case, the second channel can dominate: the dampened inference by uninformed investors outweighs the market clearing effect, and a higher A pushes the equilibrium toward unconstrained. This is why, in the first part of Lemma 4, the equilibrium is unconstrained for intermediate values of A and constrained at both extremes. When D_1 is low, aggregate price sensitivity depends primarily on fundamental investors, so the first channel dominates. A higher risk aversion A then pushes the equilibrium toward the constrained regime, yielding the monotone threshold result in the second part.

Finally, we consider how the investor composition θ affects the equilibrium regime.

Lemma 5. *There exists a constant* $\hat{D}_1 = \frac{\sigma_s}{\sigma_y(1+A\sigma_s\sigma_y)} > 0$ *such that:*

1. *If* $D_1 \leqslant \hat{D}_1$*, there exists a cutoff* $\theta^\dagger > 0$ *such that the equilibrium is constrained if and only if* $\theta \leqslant \theta^\dagger$.

2. *If* $D_1 > \hat{D}_1$*, there exist cutoffs* $0 < \underline{\theta}^\dagger \leqslant \bar{\theta}^\dagger$ *such that the equilibrium is constrained if* $\theta \in \left(0, \underline{\theta}^\dagger\right] \cup \left[\bar{\theta}^\dagger, 1\right]$.

Figure 5 illustrates Lemma 5. The effect of the mass of fundamental investors, θ, on the equilibrium regime depends on the demand elasticity of non-fundamental investors, D_1. Two forces are at work. The first is information spillover. As θ increases, more fundamental investors can trade on the score, so the score's information is revealed more strongly through the price. This information spillover induces the seller to add more noise. The second is a market composition effect. As θ increases, the non-fundamental investor share $1-\theta$ shrinks and fundamental investors account for a larger fraction of aggregate demand. This shifts the overall demand elasticity and therefore changes the firm manager's incentive to manipulate.

When θ is low, the spillover channel dominates. In this region, the seller serves all fundamental investors ($\lambda^* = 1$), so a marginal increase in θ raises the mass of informed fundamental investors

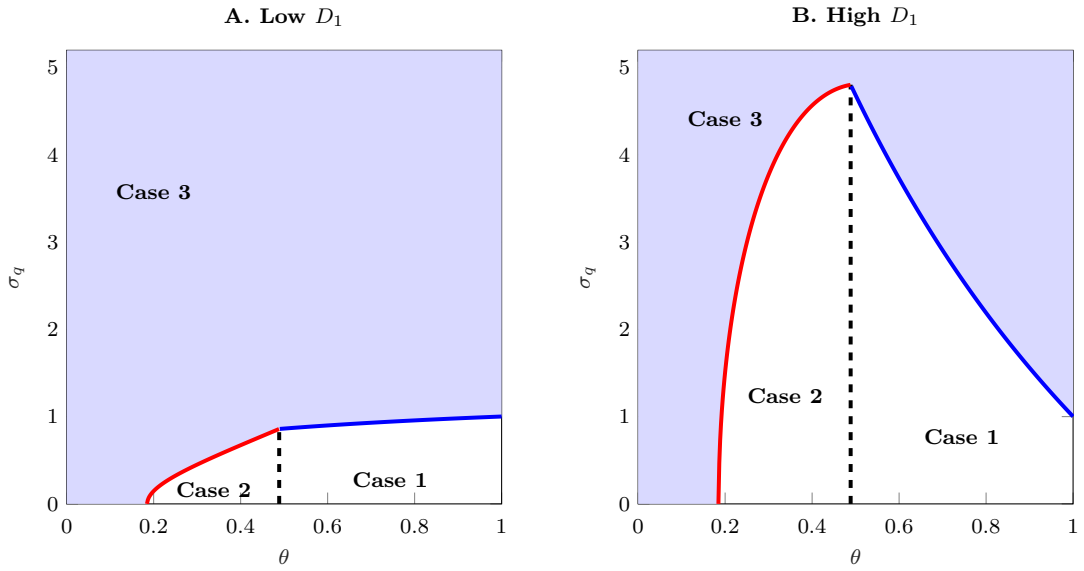


Figure 5: **Equilibrium cases in share of fundamental investors θ.** This figure plots the regime boundary separating the unconstrained region (below the boundary) from the constrained region (above the boundary) as a function of share of fundamental investors θ for two values of the non-fundamental demand elasticity D_1. The left panel sets $D_1 = 0.01$ (low D_1) and the right panel sets $D_1 = 5$ (high D_1). Within each panel, the red curve is the boundary segment corresponding to the region $\bar{\lambda} \geqslant 1$, the blue curve is the boundary segment corresponding to $\bar{\lambda} < 1$, and the vertical dashed line marks the value of θ at which $\bar{\lambda} = 1$. Labels inside each panel indicate the equilibrium regime of Theorem 1 in each region: Case 1 is the unconstrained equilibrium with partial participation, Case 2 the unconstrained equilibrium with full participation, and Case 3 the constrained equilibrium. Parameter values are $\rho = 0.9$, $c = 1.5$, $\sigma_s = 0.5$, $\sigma_y = 1.5$, $A = 1$, and $\underline{\sigma} = 0.1$.

one for one. The resulting increase in information spillover leads the seller to choose a noisier score, creating slack to absorb manipulation induced noise and pushing the equilibrium toward the unconstrained regime. As a result, the equilibrium is constrained only when θ is sufficiently low.

When θ is higher, this logic changes. The seller no longer maintains $\lambda^* = 1$. Instead, she reduces the fraction of score buyers λ^* as θ rises, keeping the total mass of informed fundamental investors, $\theta\lambda^*$, stable. This offsets the spillover effect, so the main remaining force is the change in aggregate demand elasticity driven by market composition.

The direction of this composition effect depends on D_1. If D_1 is low, non-fundamental demand is relatively inelastic, so replacing non-fundamental investors with fundamental investors makes aggregate demand more elastic. The price then responds less to the score, the firm manager's

manipulation incentive falls, and the equilibrium moves toward the unconstrained region. If D_1 is high, non-fundamental demand is highly elastic, so replacing non-fundamental investors with fundamental investors makes aggregate demand less elastic. The price then responds more strongly to the score, which strengthens the firm manager's manipulation incentive and can push the equilibrium back into the constrained region.

This explains the non-monotonicity in the second part of Lemma 5. When D_1 is high, the constrained equilibrium arises at both low and high values of θ, but for different reasons. At low θ, there are few fundamental investors, the information spillover into the price is weak, and the seller has little reason to add noise to the score. At high θ, fundamental investors dominate aggregate demand, making it less elastic and strengthening the firm manager's manipulation incentive. At intermediate values of θ, neither force is strong enough to exhaust the seller's noise absorption capacity, and the equilibrium remains unconstrained.

Taken together, Lemmas 2–5 show that manipulation is more likely to affect price informativeness when the raw signal is less informative (low ρ) and when non-fundamental demand is less price elastic (low D_1). The effects of risk aversion A and the fundamental investor share θ can be non-monotone. These results characterize *when* manipulation matters, but they are silent on *how much* manipulation distorts market efficiency once the economy is there. We turn to this question next.

3 Implications for Market Efficiency

A common intuition is that fundamental investors, who trade on information, make prices more efficient, so markets with more of them should have more informative prices. This section shows that information manipulation can overturn this intuition. In a constrained equilibrium, a larger share of fundamental investors can reduce price informativeness, while non-fundamental investors, who process no information at all, can improve it. Two features of the non-fundamental investor segment drive these results: their demand elasticity D_1 and their mass $1 - \theta$.

3.1 Investor Composition and Price Efficiency

We begin with the role of non-fundamental investor demand elasticity, which provides the foundation for understanding how changes in investor composition affect informativeness. In the baseline model, Lemma 3 established that the demand elasticity of non-fundamental investors D_1 determines whether the equilibrium is constrained: a lower D_1 makes the constrained regime more likely. Proposition 4 shows that D_1 also governs the severity of manipulation's effect on informativeness within the constrained regime.

Proposition 4. *Fix all primitives other than the demand elasticity of non-fundamental investors D_1 and suppose $\theta < 1$. Both score and price informativeness are strictly increasing in D_1 on any open interval over which the equilibrium remains constrained.*

The proposition connects the mechanism in Lemma 3 to information quality. More elastic non-fundamental demand reduces the equilibrium price sensitivity to the score, a_r, weakening the manager's incentive to manipulate and reducing the variance that manipulation adds. With the seller continuing to choose the noise floor, this reduction improves both score and price informativeness.

This result identifies a new channel through which non-fundamental investors contribute to market efficiency. Their demand elasticity affects the firm manager's manipulation incentive by limiting the price response to the score. This effect operates entirely through the equilibrium object a_r and requires no informational role for non-fundamental investors: they do not observe or process the score.

Remark 1 (Benchmark cases for D_1)**.** *If the equilibrium is unconstrained (the first two cases of Theorem 1), or if there is no manipulation ($q \equiv 0$), then both* $\mathrm{corr}(g_r, \tilde{g})$ *and* $\mathrm{corr}(p, \tilde{g})$ *are independent of D_1.*

Remark 1 highlights that the effect of D_1 on informativeness is entirely a consequence of manipulation. In the unconstrained equilibrium, the information seller retains slack in her noise choice and can offset any change in a_r caused by variation in D_1 by adjusting σ_u. This keeps the total variance of the score at its revenue maximizing target and leaves informativeness unchanged.

In the no-manipulation benchmark ($q \equiv 0$), D_1 affects the price level but has no effect on the informativeness of either the score or the price about $\tilde{g}$. Proposition 4 therefore identifies a channel that is novel to the model: it requires both the presence of manipulation ($\sigma_q > 0$) and the seller's inability to absorb the manipulation induced noise ($\sigma_u = \underline{\sigma}$).

We next turn to the effect of investor composition. Proposition 5 establishes that in the constrained equilibrium with full participation, the effect of the fundamental investor share θ on price informativeness can be non-monotone when D_1 is sufficiently large.

Proposition 5. *Let σ_q^{full} be the threshold in Proposition 3, which is independent of θ. For every $\sigma_q \geqslant \sigma_q^{\text{full}}$, there exists a cutoff $D_{\text{crit}}(\sigma_q) > 0$ (solved in the appendix) such that the effect of θ on price informativeness is as follows.*

1. *If $D_1 \leqslant D_{\text{crit}}(\sigma_q)$, $\frac{d}{d\theta}\operatorname{corr}(p, \tilde{g})\big|_{\theta=\theta_0} > 0$ for any $\theta_0 \in (0, 1)$.*
2. *If $D_1 > D_{\text{crit}}(\sigma_q)$, there exists a unique $\bar{\theta} \in (0, 1)$ such that price informativeness is uniquely maximized at $\bar{\theta}$, and*

$$\frac{d}{d\theta}\operatorname{corr}(p, \tilde{g})\Big|_{\theta=\theta_0} \begin{cases} > 0, & \theta_0 \in (0, \bar{\theta}), \\ = 0, & \theta_0 = \bar{\theta}, \\ < 0, & \theta_0 \in (\bar{\theta}, 1). \end{cases} \tag{16}$$

Proposition 5 shows that, when manipulation uncertainty is sufficiently high ($\sigma_q \geqslant \sigma_q^{\text{full}}$), the demand elasticity of non-fundamental investors determines whether the standard intuition, that more fundamental investors make prices more informative, survives manipulation. When $D_1 \leqslant D_{\text{crit}}(\sigma_q)$, it does: price informativeness rises with θ over its entire range, even though more fundamental investors strengthen the manager's incentive to manipulate.

When $D_1 > D_{\text{crit}}(\sigma_q)$, it fails: beyond $\bar{\theta}$, a larger share of fundamental investors reduces price informativeness. In this regime, all fundamental investors buy the score ($\lambda^* = 1$), and the seller chooses the minimum noise level ($\sigma_u^* = \underline{\sigma}$). An increase in θ produces two opposing effects. First, more investors trade on the score, so its information is revealed more in the equilibrium price. Second, the price is more sensitive to the score: the resulting increase in a_r strengthens the manager's

manipulation incentive and raises the manipulation-induced variance $a_r^2\sigma_q^2/c^2$. Because the seller is already at her noise floor, she cannot offset this increase. When θ is low, the informational benefit dominates, and adding fundamental investors raises price informativeness.

Whether this improvement eventually reverses depends on D_1. Within the equilibrium region covered by Proposition 4, greater non-fundamental demand elasticity improves price informativeness for a fixed investor composition by dampening the price response to the score and weakening manipulation. As θ increases, the mass of non-fundamental investors shrinks, and this stabilizing force weakens. In the limiting case $\theta = 1$, non-fundamental investors are absent, so their manipulation-reducing effect disappears and economies with different values of D_1 converge to the same price informativeness. For large D_1, informativeness at some interior θ can therefore exceed its value at $\theta = 1$, producing the hump-shaped pattern.

The cutoff $D_{\text{crit}}(\sigma_q)$ identifies when the manipulation effect reverses the overall effect of θ. At $D_1 = D_{\text{crit}}(\sigma_q)$, the marginal informational benefit of adding fundamental investors exactly offsets the informativeness loss from stronger manipulation at $\theta = 1$. If $D_1 \leqslant D_{\text{crit}}(\sigma_q)$, the informational benefit dominates for all $\theta \in (0, 1)$ and price informativeness is strictly increasing; at equality, its slope reaches zero only at the endpoint. If $D_1 > D_{\text{crit}}(\sigma_q)$, the manipulation effect eventually dominates: price informativeness rises initially, reaches its unique maximum at $\bar{\theta}$, and then declines toward the common endpoint. The non-monotonicity reflects the tension between the informational gain from greater fundamental participation and the loss of non-fundamental investors' stabilizing demand, in a regime where the seller can no longer absorb manipulation-induced noise. Figure 6 illustrates this result.

We focus on full participation in Proposition 5 for tractability. With partial participation, the seller can adjust the fraction of score buyers through the score fee even when her noise choice is constrained, complicating a complete characterization. Nevertheless, the results in Proposition IA.1 of the Internet Appendix confirm the robustness of the central role of non-fundamental demand elasticity. In constrained equilibrium, price informativeness increases with θ when D_1 is below a lower cutoff, and decreases with θ under partial participation when D_1 exceeds an upper cutoff.

Proposition 5 covers only the constrained equilibrium. How does θ affect price informativeness

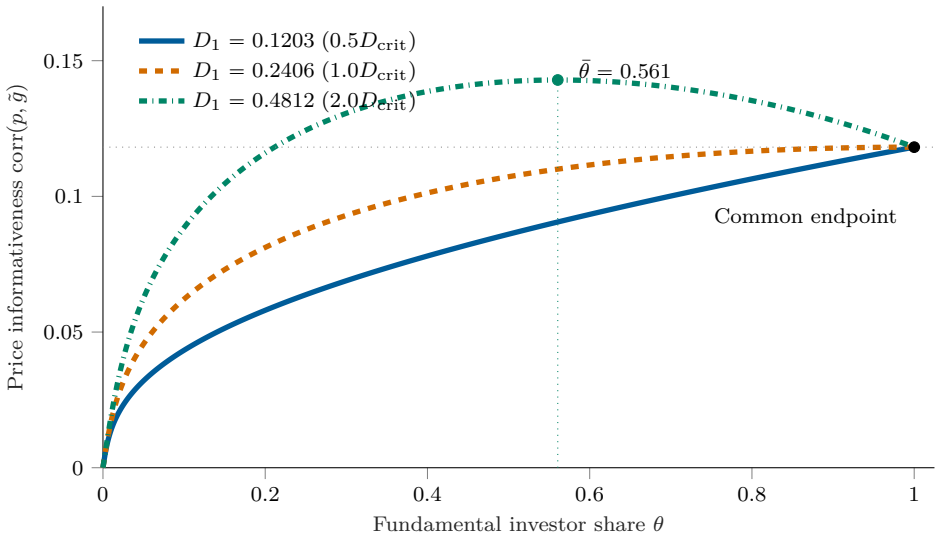


Figure 6: **Price Informativeness and Fundamental Investor Share.** The figure plots price informativeness $\text{corr}(p, \tilde{g})$ as a function of the fundamental investor share θ for three values of D_1 relative to the critical threshold D_{crit}: $D_1 = 0.5D_{\text{crit}}$ (solid), $D_1 = D_{\text{crit}}$ (dashed), and $D_1 = 2D_{\text{crit}}$ (dash-dotted). When $D_1 \leqslant D_{\text{crit}}$, price informativeness is monotonically increasing in θ. When $D_1 > D_{\text{crit}}$, price informativeness is hump-shaped, reaching its unique maximum at $\bar{\theta}$ (marked by the dot and vertical dotted line). All three curves converge to the same value at $\theta = 1$ (black dot). Parameter values are $A = 3$, $c = 1.5$, $\rho = 0.9$, $\sigma_y = 1.5$, $\sigma_s = 0.5$, $\underline{\sigma} = 0.1$, and $\sigma_q = 8$, yielding $D_{\text{crit}} = 0.2406$.

when the equilibrium is unconstrained? Proposition 2 shows that informativeness in that case coincides with the benchmark without manipulation, and the following remark characterizes its relationship with investor composition.

Remark 2. *In the unconstrained equilibrium of Theorem 1,* $\text{corr}(p, \tilde{g})$ *is independent of* θ *under partial participation* ($\lambda^* < 1$) *and decreasing in* θ *under full participation* ($\lambda^* = 1$).

Remark 2 provides a useful benchmark. In the unconstrained equilibrium with full participation, price informativeness is also decreasing in θ, but the mechanism is entirely different. In that case, all fundamental investors already buy the score ($\lambda^* = 1$), and the seller retains slack in her noise choice ($\sigma_u^* > \underline{\sigma}$). As θ rises, the increased mass of informed traders strengthens information spillover through the price. The seller responds by adding more noise to the score, which reduces both score and price informativeness. The decline in informativeness is therefore driven by the seller's strategic noise addition, not by the firm manager's manipulation.

In contrast, in Proposition 5 with $D_1 > D_{\text{crit}}(\sigma_q)$, the seller is constrained at $\sigma_u = \underline{\sigma}$ and

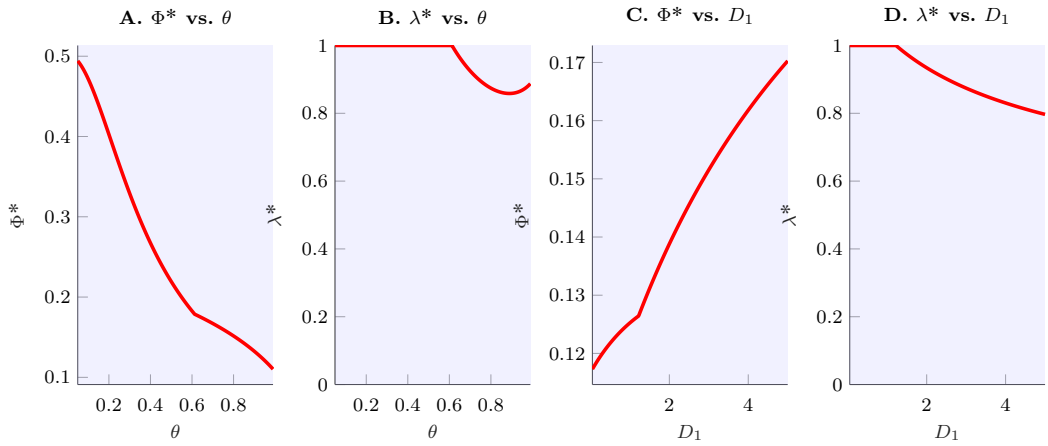


Figure 7: **Comparative Statics in θ and D_1.** The figure plots the equilibrium score fee Φ^* and the fraction of informed fundamental investors λ^* as functions of the share of fundamental investors θ (Panels A and B) and the slope of non-fundamental demand D_1 (Panels C and D). The blue shading indicates the constrained equilibrium, which prevails throughout the plotted ranges of θ and D_1. Panels B and D show full participation ($\lambda^* = 1$) at low values of θ and D_1, respectively, and partial participation ($\lambda^* < 1$) at higher values. Parameter values are $\sigma_g = 1$, $\sigma_y = 1.5$, $\rho = 0.8$, $A = 1$, $\sigma_s = 0.5$, $\sigma_q = 2.5$, $\underline{\sigma} = 0.1$, $c = 1.5$, and $D_0 = 1$. Panels A and B hold $D_1 = 3$; Panels C and D hold $\theta = 0.8$.

cannot reduce the added noise further; the decline in price informativeness is driven entirely by the firm manager's manipulation. The same qualitative result (price informativeness declining in θ) thus arises from two distinct forces: the seller's strategic noise addition in the unconstrained equilibrium with full participation and the manager's manipulation in the constrained equilibrium. The latter channel is the novel contribution of the model and operates when the seller's noise choice is constrained. Figure 7 illustrates the seller's pricing and participation choices in the constrained regime as θ (Panels A and B) and D_1 (Panels C and D) vary. The score fee declines with θ and rises with D_1. Participation is full at low values of θ and D_1 and partial at higher values.

These results shed light on the debate over the rise of benchmarking and passive investing. A common concern is that as more capital is managed passively or against a benchmark, fewer investors trade on information and prices become less efficient. When manipulation is severe enough that the information seller can no longer absorb it, our analysis suggests that this concern is incomplete in two ways.

First, non-fundamental investors affect price efficiency even though they do not trade on

information. Price-elastic demand, such as that of equal-weight index funds, target-date funds, and market makers, dampens the price response to the score and thereby weakens the firm's incentive to manipulate it. Price-inelastic demand, such as that of cap-weighted index funds and benchmarked fund managers (Pavlova and Sikorskaya, 2023), does the opposite.[9] The evidence that aggregate demand for stocks has become less price elastic (Gabaix and Koijen, 2021) is therefore relevant not only for price volatility but also for information quality.

Second, a shift toward non-fundamental investing need not reduce price informativeness. When non-fundamental demand is sufficiently elastic and fundamental investors already dominate the market, a moderate shift toward passive or benchmarked strategies weakens manipulation by more than it reduces information-based trading, and prices become more informative. When non-fundamental demand is inelastic, the conventional concern applies. Evaluating the growth of benchmarking and passive investing therefore requires asking not only how much capital they attract but also how price elastic the resulting demand is. Green investing, where ESG ratings are hard to verify and firms' incentives to greenwash vary widely, is a natural setting in which manipulation is likely to be severe; we turn to it next.

3.2 Green Investor Participation and Greenwashing

Our results have natural implications for green investing. Green investing, the practice of allocating capital toward firms with higher "greenness" to promote ESG objectives, has grown rapidly in global financial markets.[10] A distinguishing feature of this market is investors' reliance on information intermediaries for assessments of firm greenness. Unlike traditional financial data such as earnings and revenues, which are subject to established reporting standards and wide public dissemination, information about a firm's environmental attributes is difficult to verify independently and is largely produced and distributed by ESG rating agencies such as MSCI and Sustainalytics (Berg

[9] In our microfoundation (Appendix A), $D_1 = 1/(\gamma(a+b)\sigma_y^2)$, so stronger relative-performance incentives (higher b for a given a) lower D_1.

[10] According to the Global Sustainable Investment Alliance (2023), total sustainable investment assets under management reached $30.3 trillion globally in 2022, with non-U.S. markets alone recording a 20% increase relative to 2020. Bloomberg Intelligence (2024) projects these assets will surpass $40 trillion by 2030, representing over a quarter of projected global assets under management.

et al. (2022)).[11] This informational environment creates strong incentives for firms to engage in "greenwashing," that is, the costly manipulation of signals about their environmental performance to inflate the firm's reported greenness.[12]

Our baseline model provides a natural framework for analyzing the interaction between green investing and greenwashing. The key model primitives can be relabeled as follows. Investors' utility still takes the form Eq. (2), with $\tilde{y}$ decomposed as in Eq. (1). We interpret $\tilde{g}$ as the firm's true greenness and $\tilde{y}$ as a green investor's total payoff from holding the stock, which combines greenness with other firm characteristics (such as cash flow or risk) captured by the residual ϵ_y. Similar to Pástor et al. (2021), the parameter ρ measures the importance of greenness in the investor's preference. Fundamental investors are green investors, and their mass θ measures green investor participation. Informed green investors are those who purchase the ESG score from the rating agency, while uninformed green investors infer greenness from the stock price. We refer to non-fundamental investors as traditional investors. They do not value greenness; the dimensions of the firm they care about, such as cash flow, are captured by ϵ_y, about which no market participant has information. They therefore have nothing to learn from the score or the price, and their demand takes the form in Eq. (3).[13] Under this interpretation, the score g_r corresponds to the ESG score produced by a rating agency, the firm manager's signal manipulation δ captures greenwashing activity, the uncertainty σ_q reflects the heterogeneity in firms' incentives to greenwash, and the information seller is an ESG rating agency that designs and prices the score to maximize profit. Because these relabelings do not alter any equation or equilibrium condition, all results for the baseline model apply directly to the green investing setting. We now discuss implications for markets with green investing.

First, the manipulation irrelevance result (Proposition 2) implies that greenwashing, even if widespread, need not reduce the informativeness of ESG scores or stock prices. What matters is not how much firms greenwash on average but how predictable their greenwashing is. When firms'

[11] According to a 2022 survey by management consultancy Opimas, just four ESG data providers account for nearly 70% of the market, with MSCI alone commanding approximately 31%.

[12] A prominent example is the Volkswagen emissions scandal, in which the company admitted to rigging emissions tests in over eleven million diesel vehicles worldwide, falsely marketing them as "clean diesel," resulting in more than $30 billion in global penalties.

[13] For example, if a traditional investor has CARA utility over the firm's cash flow, her demand is linear in the price, as in Eq. (3), with slope D_1 equal to the inverse of the product of her risk aversion and the cash-flow variance. More risk-tolerant traditional investors thus correspond to a higher D_1.

incentives to greenwash are similar (low σ_q), investors discount the average inflation in the ESG score, and the rating agency offsets the remaining manipulation-induced noise by reducing its own added noise σ_u, so that score and price informativeness are the same as in the benchmark without greenwashing. Only when $\sigma_q \geqslant \sigma_q^{\dagger}$ can greenwashing make the ESG score and the stock price less informative about the firm's true greenness.

Second, a central goal of green investing is to improve price discovery about firms' environmental attributes, yet expanding green investor participation can undermine it. By Proposition 5, when firms' greenwashing incentives are sufficiently heterogeneous ($\sigma_q \geqslant \sigma_q^{\text{full}}$) and traditional investors' demand is sufficiently price elastic ($D_1 > D_{\text{crit}}(\sigma_q)$), price informativeness about greenness, $\text{corr}(p, \tilde{g})$, is hump-shaped in θ. When θ is low, more green investors trade on the ESG score, and their aggregate demand impounds more information about $\tilde{g}$ into the stock price. When θ is high, green investors replace traditional investors, whose elastic demand dampens the price response to the ESG score. The price then becomes more sensitive to the score, the firm manager's incentive to greenwash strengthens, and further increases in θ reduce price informativeness. Under these conditions, the model predicts an interior level of green investor participation that maximizes price informativeness about true greenness.[14] For regulators, the result suggests that policies expanding the green investor base without strengthening anti-greenwashing enforcement may reduce price informativeness about true greenness.

4 Extensions

In this section, we examine four extensions to the baseline model. The extensions are designed to shed further light on the implications of information manipulation in the following settings: real investment, mandate investing, endogenous verification, and multidimensional score design. Collectively, these cases illustrate the broader applicability of the mechanisms highlighted in the baseline framework.

[14]This prediction contrasts with Goldstein et al. (2026), in which price informativeness increases monotonically with the share of green investors, because their model does not feature endogenous information manipulation. In our model, when $\sigma_q \geqslant \sigma_q^{\text{full}}$ but traditional investors' demand is relatively inelastic ($D_1 \leqslant D_{\text{crit}}(\sigma_q)$), price informativeness also rises with θ throughout.

4.1 Information Manipulation and Real Investment

In the baseline model, the firm manager can boost the stock price only through signal manipulation, which distorts the information content of the score without creating real value and is therefore a pure deadweight loss. In practice, firms can also raise their stock price by making real investments that improve firm quality, such as expanding productive capacity, investing in research and development, or, in the case of green investing, investing in technology that reduces carbon emissions. An important distinction is that information manipulation (distorting financial disclosures, engaging in fraudulent accounting, or misrepresenting research results) is typically an ex-post action, while real investment is typically an ex-ante action.

A natural concern is that manipulation may crowd out real investment: when the firm manager can inflate the stock price cheaply by distorting the score, he may have less incentive to create genuine value. To examine whether and when such crowding out occurs, we add an ex-ante investment stage to the baseline model. Because the manager's manipulation and investment costs are separable, crowding out cannot arise from a direct trade-off in his own problem; any interaction must operate through the information market, that is, through how manipulation affects the score's informativeness and hence the price response to the score. We show that crowding out is regime-dependent: cheaper manipulation has no effect on real investment in the unconstrained equilibrium, where the information seller absorbs the manipulation-induced noise, but reduces real investment in the constrained equilibrium, where she can no longer do so.

Ex Ante investment. At a new stage $t = -1$, the firm manager searches for an investment project of quality $I \in \mathbb{R}$, incurring effort cost $\frac{c_I}{2} I^2$ with $c_I > 0$. The investment raises the value of $\tilde{g}$ to $\tilde{g} + I$, so that the payoff decomposition becomes

$$\tilde{y} = \bar{y} + \rho\sigma_y\big(\tilde{g} + I\big) + \epsilon_y. \tag{17}$$

The choice of I is private information of the manager. At time 0, market participants form a common belief $\tilde{I}$; in equilibrium, the consistency condition $\tilde{I} = I$ holds. The continuation game is

otherwise identical to the baseline: the manager privately observes their manipulation incentive q, which is realized after investment I is chosen, and chooses manipulation δ. The realized score now incorporates the investment,

$$g_r = \tilde{g} + I + u + \delta. \tag{18}$$

Real investment thus creates genuine fundamental value in $\tilde{y}$, while signal manipulation only distorts the information environment.

Continuation equilibrium. Because I shifts only means and leaves all conditional variances unchanged, the structure of the continuation equilibrium carries over from the baseline.

Lemma 6. *For any market belief $\tilde{I}$, the time-0 continuation equilibrium is characterized by Theorem 1. The case boundaries and the equilibrium objects $\lambda^*, \Phi^*, \sigma_u^*, \delta^*$ are independent of $\tilde{I}$, and the equilibrium price takes the form*

$$p = a_0(\tilde{I}) + a_r^* g_r + a_s^* \tilde{s}, \tag{19}$$

where the loadings (a_r^*, a_s^*) coincide with their baseline values; only the intercept a_0 depends on $\tilde{I}$.

The market belief $\tilde{I}$ shifts every investor's conditional mean of $\tilde{y}$ by $\rho\sigma_y\tilde{I}$ but changes no conditional variance. Since the information seller's design problem, the price loadings, and the case boundaries all depend on conditional variances (and not on conditional means), they are identical to their baseline values. Only the price intercept responds to $\tilde{I}$. Intuitively, the ex-ante investment genuinely raises firm value but leaves the information environment untouched.

The firm manager's optimal investment choice is characterized in the following result.

Lemma 7. *The firm manager's time -1 problem is equivalent to*

$$\max_I \ \bar{q}\,\mathbb{E}[p] - \frac{c_I}{2} I^2, \tag{20}$$

with unique optimum

$$I^* = \frac{\bar{q}}{c_I}\, a_r^*. \tag{21}$$

Because I is unobservable, its marginal benefit runs entirely through the score's price pass-

through a_r^*: any force that dampens how strongly the price responds to the score also dampens the firm's real investment incentive.

Manipulation and real investment. We now turn to how the firm manager's information manipulation affects real investment.

Proposition 6. *In the unconstrained equilibrium,* $\frac{dI^*}{dc} = 0$. *In the constrained equilibrium,* $\frac{dI^*}{dc} > 0$.

The proposition uncovers a regime-dependent interaction between ex-post manipulation and ex-ante real investment that runs through the information seller's design. By (21), the firm manager's investment incentive is governed entirely by the price pass-through a_r^*: anything that weakens how strongly the price responds to the score also weakens the incentive to create genuine value. The manipulation cost c affects a_r^* only when the seller can no longer absorb additional manipulation noise through her own noise choice. In the unconstrained regime she retains that slack, so a_r^* (and therefore I^*) is independent of c, and manipulation and investment activities are independent. In the constrained regime, the slack is gone: a fall in c raises manipulation δ^*, increases the noise in g_r, lowers a_r^*, and thereby reduces I^*. Cheaper manipulation thus does double damage in this regime, simultaneously inflating signal distortion and discouraging real value creation.

The crowding out arises because investment and manipulation enter the score identically (see (18)), so investors cannot tell genuine improvements from manipulated ones. Anticipating manipulation, the market discounts the entire score, including its legitimate component, and the manager invests less because the price rewards his investment less.

This result points to a real benefit of curbing manipulation beyond improving price informativeness. In the constrained regime, measures that raise the cost of manipulation, such as stricter auditing, third-party verification, or penalties for greenwashing, make the price more responsive to the score and thereby strengthen firms' incentives to invest; for example, curbing greenwashing can encourage genuine emissions reductions. The information seller does not internalize this benefit, because real investment leaves her design problem (including her profit) unchanged (Lemma 6). As Section 4.3 shows, even when the seller can invest in verification herself, she never restores informativeness to its unconstrained level. Thus, the effect on real investment strengthens the case

for policies that curb manipulation directly.

4.2 Mandate Investors

In the baseline model, the score provided by the information seller is valuable because it helps fundamental investors forecast asset payoffs. In practice, however, some investors may value the score itself, even if it is noisy or biased (Baghai et al., 2023). This arises when investors face investment mandates that depend directly on the score, or when fund managers' compensation contracts depend on a score-based benchmark portfolio. For example, certain institutional investors are permitted to invest only in corporate bonds with an investment grade rating, and an ESG fund may be required to hold a portfolio that exceeds a minimum ESG score.

In this section, we use a variation of the baseline model to examine how the presence of investment mandates changes the market equilibrium and its implications for market efficiency. We extend the baseline model by introducing mandate-sensitive non-fundamental demand. The information seller, the firm manager, all fundamental investors, and the timing remain exactly as in Section 2. The only modification is to the non-fundamental investor demand function.

In the baseline, each non-fundamental investor's demand for the risky asset is $D(p) = D_0 - D_1 p$, which depends only on the price. Here, we replace it with

$$D(p, g_r) = D_0 - D_1 p + D_2 g_r, \tag{22}$$

where $D_2 > 0$ is a mandate sensitivity parameter and g_r is the realized score. It captures the extent to which non-fundamental investors hold more of the stock when its reported score is higher, independently of what the score reveals about $\tilde{g}$. For example, index funds that overweight stocks above an ESG score threshold, or ETFs whose inclusion rules are tied to reported ratings, increase their holdings when the reported score rises, even if the score's information about the underlying attribute has not changed. Many such rules involve thresholds; we adopt the linear specification in (22) as a tractable approximation. Because these investors respond to the score level rather than to its informational content, we model them as part of the non-fundamental investor population.When

$D_2 = 0$, the model reduces to the baseline.

In Appendix A, we derive this demand from delegated portfolio choice, building on the benchmarking model of Pavlova and Sikorskaya (2023).[15] There, fund managers are paid partly on absolute performance and partly on performance relative to a benchmark whose holding of the stock rises with the reported score. D_2 equals the share of the contract's performance pay tied to the benchmark times the sensitivity of the benchmark's holding to the score (Proposition A.1). Mandate demand is therefore stronger when managers' pay is tied more closely to the benchmark, or when the benchmark tilts more heavily toward high-score firms.

The score in this model serves two distinct roles. For fundamental investors, the score is a statistical signal: its value lies in reducing uncertainty about $\tilde{g}$ and hence about $\tilde{y}$. For non-fundamental investors who are now mandate-sensitive, the score is a compliance input: its value lies in determining portfolio allocations, independently of the score's precision. We assume that the information seller can price discriminate between these two uses. As in the baseline, fundamental investors' decision to purchase the score at a price Φ depends on the score's informativeness. Mandate-sensitive non-fundamental investors purchase access to the score level at an exogenous fixed fee that does not depend on the seller's design choices (σ_u, λ).[16] Since the revenue from non-fundamental investors is a constant, the seller's optimization problem is equivalent to maximizing revenue from fundamental investors alone, and the formal structure of the seller's problem is identical to the baseline.

Assumption 1. $D_1 > \frac{\rho}{\sigma_y} D_2$.

Assumption 1 guarantees that, as in the baseline, the price becomes less sensitive to the score when the score becomes noisier. This property keeps manipulation self-limiting: the noise that

[15] Pavlova and Sikorskaya (2023) frame their analysis in terms of benchmarking rather than mandates, but the two are closely connected: benchmark relative compensation is one of the principal contractual mechanisms through which a mandate is implemented. Formally, the benchmark tracking component of the optimal portfolio pins a fraction of the manager's position to the prescribed benchmark holding, independently of the manager's own assessment of payoffs. This mechanical component of demand is exactly what we mean by a mandate.

[16] For mandate investors, the score is valuable because a regulatory or contractual rule references it, not because of what it reveals about fundamental information. Their willingness to pay therefore does not depend on the score's precision. MSCI, for instance, licenses the ESG indexes built from its ESG Ratings to the funds that track them, charging a fee based primarily on those funds' assets under management; the fee is priced on use and scale, not on the accuracy of the underlying ratings.

manipulation adds to the score lowers a_r, and with it the manager's incentive to manipulate. Investment mandates put this property at risk. When the score becomes noisier, fundamental investors respond less both to the score and to the price. Mandate investors, by contrast, respond to the score without regard to its informativeness, so their demand shift per unit of score is unchanged. The two weaker responses of fundamental investors push the score's price impact in opposite directions. Because they trade less on the score, a given change in the score shifts aggregate demand by less, which lowers its price impact. But because fundamental investors also trade less in response to the price, the market is less able to absorb demand shifts, so the same shift in demand, including the unchanged shift from mandate investors, requires a larger price change, which raises the score's price impact.

The net effect on the price's sensitivity to the score is thus ambiguous. When D_2/D_1 is large enough, the second effect dominates: the demand shift per unit of score comes mainly from mandate investors and is unaffected by noise, whereas the demand response to price that absorbs it comes mainly from fundamental investors and shrinks with noise, so a noisier score moves the price more rather than less. Assumption 1 rules this out.[17] The following lemma characterizes how the uncertainty in the firm manager's manipulation incentive determines the equilibrium regime, paralleling Proposition 3 in the baseline model.

Lemma 8. *There exist thresholds $\sigma_{q,M}^{\text{full}} \geqslant \sigma_{q,M}^{\dagger} \geqslant 0$, with $\sigma_{q,M}^{\text{full}}$ independent of D_2 and $\sigma_{q,M}^{\dagger} = \sigma_q^{\dagger}$ when $D_2 = 0$, such that the equilibrium is unconstrained if and only if $\sigma_q < \sigma_{q,M}^{\dagger}$. Moreover, if $\sigma_q \geqslant \sigma_{q,M}^{\text{full}}$, the information seller chooses $\lambda^* = 1$.*

The qualitative structure of Lemma 8 is the same as in the baseline: when σ_q is low, the seller retains slack in her noise choice and the equilibrium is unconstrained; when σ_q is high, the constraint

[17] Regarding the ratio ρ/σ_y, intuitively, for a fundamental investor, the information that drives the response to the score also drives the response to the price, so the two move in lockstep at roughly the rate ρ/σ_y. Mandate demand carries no such link; the assumption requires that its ratio of price response to score response, D_1/D_2, nonetheless exceed this rate. To see the mechanics in more detail, suppose all fundamental investors observe the score ($\lambda = 1$); the appendix treats the general case. Let $r_1 \equiv \text{Cov}(\tilde{g}, g_r)/\text{Var}(g_r)$ measure the score's informativeness, which falls as the score becomes noisier, and let $V_1 \equiv \text{Var}(\tilde{y} \mid g_r) = \sigma_y^2(1 - \rho^2 r_1)$. A fundamental investor demands $(\mathbb{E}[\tilde{y} \mid g_r] - p)/(AV_1)$ with $\mathbb{E}[\tilde{y} \mid g_r] = \bar{y} + \rho\sigma_y r_1(g_r - \mathbb{E}[g_r])$, so this demand rises by $\rho\sigma_y r_1/(AV_1)$ per unit of score and falls by $1/(AV_1)$ per unit of price. The two fundamental terms are linked by the identity $1/V_1 = 1/\sigma_y^2 + (\rho/\sigma_y) \cdot \rho\sigma_y r_1/V_1$: a fundamental investor's price sensitivity equals a noise-invariant floor plus ρ/σ_y times the investor's score sensitivity. Market clearing then gives $1/a_r = \rho/\sigma_y + \Xi z$, where $z \equiv -a_s/a_r$ increases with noise and $\Xi \equiv \theta/(A\sigma_y^2) + (1-\theta)\big(D_1 - (\rho/\sigma_y)D_2\big)$; the assumption ensures $\Xi > 0$, so a_r falls with noise (see the proof of Lemma 8).

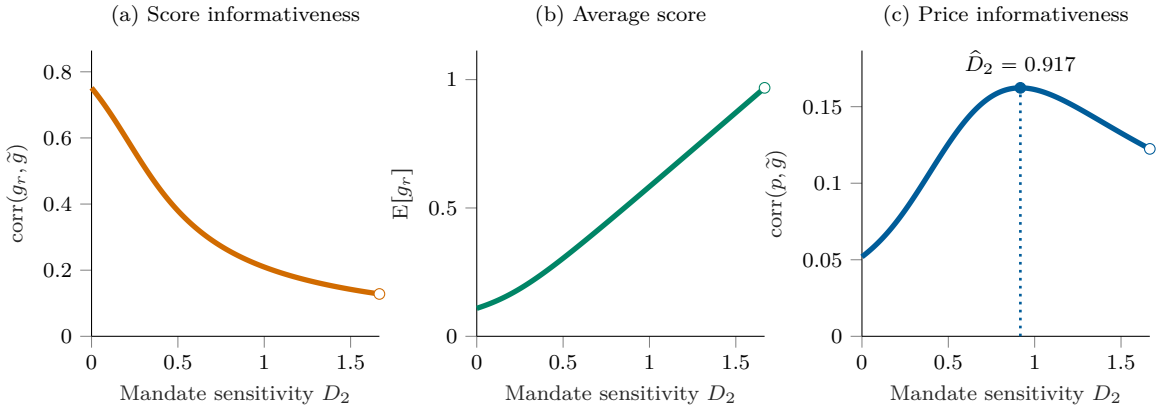


Figure 8: **Mandate Sensitivity and Informativeness.** The figure plots score informativeness $\mathrm{corr}(g_r, \widetilde{g})$ (Panel A), the average score $\mathrm{E}[g_r]$ (Panel B), and price informativeness $\mathrm{corr}(p, \widetilde{g})$ (Panel C) as functions of the mandate sensitivity D_2 in the constrained equilibrium with $\lambda^* = 1$ and $\sigma_u^* = \underline{\sigma}$. Score informativeness is strictly decreasing and the average score is strictly increasing in D_2. Price informativeness is hump-shaped, reaching its unique maximum at $\widehat{D}_2$. Open circles denote the limits as $D_2 \uparrow \sigma_y D_1/\rho$. Parameter values are $A = 3$, $c = 1.5$, $\rho = 0.9$, $\sigma_y = 1.5$, $\sigma_s = 2$, $\theta = 0.5$, $\underline{\sigma} = 0.1$, $D_1 = 1$, $\bar{q} = 1$, and $\sigma_q = 8$.

$\sigma_u \geqslant \underline{\sigma}$ binds and the equilibrium becomes constrained.

We now examine how mandate sensitivity affects market efficiency in the constrained regime.

Proposition 7. *Suppose $\theta < 1$ and $\sigma_q \geqslant \sigma_{q,M}^{\text{full}}$, then:*

1. *Score informativeness $\mathrm{Corr}(g_r, \widetilde{g})$ is strictly decreasing in D_2.*
2. *The average score $\mathbb{E}[g_r]$ is strictly increasing in D_2.*
3. *There is a unique $\widehat{D}_2 \in \left(0, \frac{\sigma_y}{\rho} D_1\right)$ such that price informativeness $\mathrm{Corr}(p, \widetilde{g})$ is strictly increasing in D_2 on $[0, \widehat{D}_2]$ and strictly decreasing in D_2 on $\left[\widehat{D}_2, \frac{\sigma_y}{\rho} D_1\right)$.*

Moreover, as $\sigma_q \to \infty$ with the other primitives fixed, $\widehat{D}_2 \to 0$.

Proposition 7 shows that, when uncertainty about manipulation incentive σ_q is sufficiently high, stronger investment mandates reduce score informativeness and inflate the average score, while price informativeness first increases and then decreases. Figure 8 illustrates these comparative statics. A higher D_2 makes mandate-sensitive non-fundamental investors respond more strongly to the score, raising the price sensitivity a_r. Because the manager's optimal manipulation is $\delta^* = a_r q/c$, a higher

a_r increases both average manipulation and the noise it adds to the score. Since the seller's noise choice is already at its lower bound, she cannot offset this additional noise: score informativeness falls and the average score rises. At the same time, stronger mandate demand makes the price more responsive to the score relative to noisy asset supply. At low D_2, this stronger transmission of score information into prices outweighs the deterioration in score quality, so price informativeness improves. Beyond the turning point $\widehat{D}_2$, the deterioration in score quality dominates and price informativeness declines. As $\sigma_q \to \infty$, $\widehat{D}_2 \to 0$, so the initial region in which stronger mandates improve price informativeness vanishes.

We focus on full participation in Proposition 7 for tractability. In a constrained equilibrium with partial participation, the information seller can adjust λ^* as D_2 changes, which complicates the comparative statics. The key effects nevertheless extend to partial participation: the Internet Appendix provides sufficient conditions under which, for sufficiently small supply noise, a local increase in D_2 lowers score informativeness, raises the average score, and lowers price informativeness.

This result highlights a fundamental tension in score-based investment mandates. The mandate is intended to align portfolios with the attribute $\tilde{g}$, but by making non-fundamental demand respond to the reported score, it strengthens the firm manager's incentive to manipulate, thereby degrading the very information on which the mandate relies. When manipulation uncertainty is high, the mandate thus undermines its own informational foundation.

The role of mandate investors contrasts with the baseline mechanism in Proposition 4, where more elastic non-fundamental demand can improve informativeness by weakening manipulation incentives. The parameter D_1 dampens a_r by absorbing price pressure, whereas D_2 amplifies a_r by adding score-driven demand. Mandate investors thus affect informativeness through two opposing channels, and the net effect depends on which one dominates.

We next turn to the unconstrained equilibrium, in which the seller can adjust her noise level in response to stronger mandates.

Proposition 8. *When $\theta < 1$, throughout the unconstrained equilibrium segment, each of*

$$\mathrm{Corr}(g_r, \tilde{g}), \qquad \mathrm{Corr}(p, \tilde{g}), \qquad \mathbb{E}[g_r]$$

is strictly increasing in D_2. When D_2 increases, the equilibrium can change from unconstrained to constrained at most once, and cannot subsequently return to an unconstrained equilibrium.

In the unconstrained equilibrium, stronger mandate demand induces the seller to provide a more precise score. The seller normally adds noise to limit information leakage through fundamental investors' trades, thereby preserving the value of purchasing the score. Mandate investors, however, respond to the reported score regardless of its precision. A higher D_2 therefore makes added noise less effective at curbing information transmission into prices, which incentivizes the seller to reduce noise so that the score remains valuable relative to the free information in the equilibrium price. Although stronger mandates also raise a_r and hence manipulation-induced noise, the seller has room to adjust: at the optimum, she reduces her own noise by more than the increase in manipulation noise, so score informativeness rises. Price informativeness rises as well, because the more precise score strengthens the price's response to fundamental information relative to supply noise. At the same time, the higher a_r increases average manipulation, so $\mathbb{E}[g_r]$ rises even as the score becomes more informative.

Taken together, Proposition 7 and Proposition 8 show that the effects of mandate demand on market informativeness depend on the equilibrium regime. Throughout the unconstrained regime, a higher D_2 improves both score and price informativeness because the seller reduces her added noise by more than the increase in manipulation noise. In the constrained equilibrium with $\lambda^* = 1$ that prevails at sufficiently high manipulation uncertainty, the seller can no longer make this adjustment. Stronger mandates then reduce score informativeness, while price informativeness initially rises as the score is transmitted more strongly into prices and falls once D_2 is high enough. Score inflation occurs in both regimes because stronger mandates increase average manipulation. Inflated scores can therefore accompany either an improvement or a deterioration in information quality, depending on the seller's ability to adjust noise and the strength of mandate demand.

The comparative statics in D_2 map directly into the design of fund managers' compensation and benchmarks. As discussed after (22), D_2 is the share of the contract's performance pay tied to the benchmark times the sensitivity of the benchmark's holding to the score, whereas D_1 depends

on the contract only through its total weight on performance, absolute and relative combined.[18] Raising D_2 while holding D_1 fixed, as in Proposition 7, therefore corresponds to either of two changes: shifting pay from absolute toward benchmark-relative performance while keeping the total performance weight unchanged, which raises the benchmarking intensity emphasized by Pavlova and Sikorskaya (2023), or adopting a benchmark that tilts more steeply toward high-score assets, such as an ESG-tilted index.

Whether tying fund managers more tightly to score-based benchmarks is desirable depends on the regime. When the firm manager's manipulation incentive is relatively predictable and the equilibrium is unconstrained, tighter benchmarking improves both score and price informativeness (Proposition 8), although it inflates scores and may eventually push the equilibrium into the constrained regime. When the manipulation incentive is highly uncertain ($\sigma_q \geqslant \sigma_{q,M}^{\text{full}}$), tighter benchmarking makes scores less informative and more inflated, and beyond $\widehat{D}_2$ makes prices less informative as well (Proposition 7). Because $\widehat{D}_2 \to 0$ as $\sigma_q \to \infty$, the range of benchmarking intensities that improve price informativeness is narrowest precisely when manipulation incentives are most uncertain. For asset owners and regulators who design score-based benchmarks, the appropriate strength of the link to the score therefore depends on how predictable firms' manipulation incentives are: a strong link is benign when these incentives are predictable, but should be kept modest, or paired with measures that curb manipulation, when they are highly uncertain.

4.3 Endogenous Verification

The preceding extensions point to the value of curbing manipulation: it can encourage real investment (Section 4.1), and it matters for the design of score-based mandates (Section 4.2). This section examines whether the market itself provides the right incentives to curb manipulation, without outside intervention. The natural candidate is the information seller, whose score is degraded by manipulation. Our baseline model shows that the information seller optimally limits score informativeness to maximize revenue. One might suspect that this limit reflects the seller's toolkit

[18] Specifically, $D_1 = 1/(\gamma(a+b)\sigma_y^2)$ and $D_2 = \omega_g\, b/(a+b)$, where a and b are the contract's weights on absolute and benchmark-relative performance, ω_g is the sensitivity of the benchmark's holding to the score, and γ is the fund manager's risk aversion; see Proposition A.1.

rather than her incentives: she can add noise to the score, but she has no way to remove the noise created by the firm's manipulation. A natural question is then: if the seller could invest in verification, like fraud detection or auditing, that raises the manager's cost of manipulation, would she use it to curb manipulation, and would score and price informativeness improve in equilibrium?

This section shows that the effect of giving the information seller a verification technology is limited: neither score nor price informativeness can exceed its unconstrained level, the level the seller targets when the noise floor does not bind. The ceiling reflects the seller's incentives rather than her toolkit: the unconstrained target already maximizes her revenue, so a more informative score has no value to her. This is the same logic that underlies manipulation irrelevance (Proposition 2). When the baseline equilibrium is unconstrained, the seller does not verify at all; when it is constrained, she may verify, but never enough to reach the unconstrained level. Moreover, the seller does not internalize the benefits of curbing manipulation that accrue to others, such as higher real investment in the constrained regime. The information seller's incentive to curb manipulation is therefore limited, leaving a potential role for outside intervention.

We extend the baseline model by giving the seller one additional tool. Before choosing the score fee and noise level, she publicly commits to a verification level $v \geqslant 0$ at cost $\psi(v)$. Verification raises the manager's manipulation cost from c to $c(v)$, with $c(0) = c > 0$ and $c'(v) > 0$, while the verification cost satisfies $\psi(0) = 0$, $\psi'(v) > 0$, and $\psi''(v) \geqslant 0$. Both $c(\cdot)$ and $\psi(\cdot)$ are twice continuously differentiable.

Given any verification level v, the continuation equilibrium is simply the baseline equilibrium with manipulation cost $c(v)$. In particular, setting $v = 0$ recovers the baseline model. The seller's gross revenue at verification level v is

$$\mathcal{R}(v) \equiv \max_{\lambda \in [0,1],\, \sigma_u \geqslant \underline{\sigma}} \theta \lambda \Phi(\lambda, \sigma_u; c(v)), \tag{23}$$

where $\Phi(\lambda, \sigma_u; c)$ is the score fee of Lemma 1, written with its dependence on the noise level and the manipulation cost made explicit. She chooses v to maximize net profit:

$$v^* \in \arg\max_{v \geqslant 0} \big\{ \mathcal{R}(v) - \psi(v) \big\}. \tag{24}$$

We use "unconstrained levels" to refer to the baseline model's unconstrained benchmarks for score and price informativeness (even when those benchmarks cannot be attained).

Proposition 9. *If the baseline equilibrium is unconstrained or at the regime boundary, the seller chooses $v^* = 0$, and the equilibrium is identical to the baseline. Otherwise, the equilibrium with endogenous verification remains constrained, and both score and price informativeness are below their unconstrained baseline levels.*

The first claim of Proposition 9 follows from manipulation irrelevance: with an unconstrained baseline, the seller has slack in her noise choice, $\sigma_u^* > \underline{\sigma}$, and informativeness is already at her preferred target. Verification would lower manipulation noise, but she would simply raise σ_u to keep total noise where she wants it. Verification therefore brings cost without any revenue benefit, so she chooses $v^* = 0$.

The second claim covers the strictly constrained case, where the noise floor binds ($\sigma_u^* = \underline{\sigma}$) at $v = 0$. Here verification can raise gross revenue by reducing manipulation noise, so the seller may choose $v^* > 0$. Even then, she stops strictly short of the unconstrained levels. At the unconstrained target, total noise is at the level that maximizes gross revenue, so the first-order effect of a small change in total noise on gross revenue is zero. A small cut in verification therefore costs revenue only at second order but saves verification cost at first order. This conclusion holds for any $c(\cdot)$ and $\psi(\cdot)$ satisfying our assumptions. It requires that verification have a strictly positive marginal cost, $\psi'(v) > 0$, but does not otherwise depend on how effective or how costly verification is.

Taken together, these results place an upper bound on what verification can achieve. This bound has a direct implication for a policymaker or social planner seeking to improve score and price informativeness. Simply equipping the seller with a verification tool is not sufficient; policies must also address her incentives or her discretion over score design. Verification can nevertheless have social value even when informativeness is unchanged. If manipulation costs represent real resource use, verification reduces wasteful manipulation in the unconstrained regime, a benefit the seller does not internalize. A planner may then favor positive verification when these resource savings exceed its cost, even though the seller chooses none.

The source of this shortfall is that curbing manipulation partly produces a public good. A more

informative score makes the price more informative, which benefits investors who do not buy the score and, as Section 4.1 shows, the firm's real decisions. The seller, however, can charge only for the score, and the information that leaks into the price erodes what buyers are willing to pay for it. The same logic shapes the effect of outside enforcement. Suppose that a regulator or auditor, rather than the seller, raises the cost of manipulation. In the unconstrained regime, the seller responds by adding noise to keep the total noise in the score at her preferred level, so outside enforcement leaves score and price informativeness unchanged and only reduces wasteful manipulation. In the constrained regime, the seller does not offset it, and outside enforcement raises informativeness. Raising informativeness in the unconstrained regime therefore requires imposing constraints on the seller's design itself, for example, through precision standards.

Our results also indicate where intervention is especially valuable. Among other factors (Section 2.3), the constrained regime arises when investors are highly uncertain about the firm manager's manipulation incentive, for example about how strongly a firm is motivated to greenwash, or about how strongly a manager wants to boost the stock price ahead of equity issuance or earnings announcements. Such events also raise the average incentive to manipulate, but what matters for the regime is the uncertainty about it. In the constrained regime, the seller may verify but stops short of the unconstrained levels, cheaper manipulation can crowd out real investment, and score-based mandates can reduce score and price informativeness. Efforts to curb manipulation are therefore especially valuable in settings where manipulation incentives are highly uncertain.

4.4 Multidimensional Score Design

In the baseline model, the information seller observes a single signal about $\tilde{g}$ and controls only the noise level σ_u. In practice, information intermediaries such as rating agencies and data vendors aggregate multiple signals when constructing a score. When the firm manager can manipulate each signal independently, the seller's choice of aggregation weights affects both the intrinsic noise in the score and the firm manager's effective cost of manipulation. Like verification in Section 4.3, the weights therefore give the seller a way to raise the cost of manipulation; unlike verification, they are costless to choose but also change the score's intrinsic noise.

This extension examines how the seller optimally designs the score in this richer environment. The extension delivers two messages. First, once signals can be manipulated, the most informative score differs from the statistically optimal one. The weights that minimize intrinsic noise, the standard forecasting solution, can place too much weight on signals that are precise but easy to manipulate, because the weight on a signal is itself what invites manipulation of it. This is a form of Goodhart's law (Hennessy and Goodhart, 2023): a signal's statistical properties no longer determine its usefulness once it becomes a target. Second, a profit-maximizing seller need not produce the most informative score. When the equilibrium is unconstrained, she deliberately limits informativeness, and many different weight vectors deliver the same profit, so sellers using the same inputs can produce different scores.

Setup There are N signals about $\tilde{g}$. In the absence of manipulation, signal i equals $\tilde{g} + u_i$ for $i = 1, \ldots, N$, where $\mathbf{u} = (u_1, \ldots, u_N)^\top \sim N(\mathbf{0}, \Sigma_u)$. We assume that no two elements of $\mathbf{u}$ are perfectly correlated with each other. As in the baseline, the firm manager observes the scalar manipulation motive $q \sim N(\bar{q}, \sigma_q^2)$ and can manipulate signal i by δ_i at cost $\frac{c_i}{2}\delta_i^2$, where $c_i > 0$. The information seller chooses a weight vector $\mathbf{w} = (w_1, \ldots, w_N)^\top$ from the simplex

$$\mathcal{W} = \left\{ \mathbf{w} \in \mathbb{R}_+^N : \mathbf{1}^\top \mathbf{w} = 1 \right\},$$

and sells the resulting score at fee Φ. The score is a linear combination of the manipulated signals:

$$g_r = \sum_{i=1}^{N} w_i(\tilde{g} + u_i + \delta_i) = \tilde{g} + \sum_{i=1}^{N} w_i u_i + \sum_{i=1}^{N} w_i \delta_i. \tag{25}$$

The weight vector $\mathbf{w}$ determines two sufficient statistics that govern the equilibrium: the intrinsic noise variance

$$\sigma_u^2(\mathbf{w}) \equiv \mathbf{w}^\top \Sigma_u \, \mathbf{w}, \tag{26}$$

and the effective manipulation cost $c(\mathbf{w})$ defined by

$$\frac{1}{c(\mathbf{w})} \equiv \sum_{i=1}^{N} \frac{w_i^2}{c_i}. \tag{27}$$

All other primitives, including the timing, investor side, firm side, participation decision, and equilibrium concept, are identical to those in the baseline model (Section 2).

Equilibrium characterization For any fixed $(\lambda, \mathbf{w})$, the firm manager's optimal manipulation of signal i is $\delta_i^* = (w_i/c_i)\, a_r\, q$, where a_r is the price sensitivity to the score. Substituting, the score becomes

$$g_r = \tilde{g} + \sum_{i=1}^{N} w_i u_i + \frac{a_r}{c(\mathbf{w})}\, q. \tag{28}$$

This expression is isomorphic to the baseline score with intrinsic noise variance $\sigma_u^2(\mathbf{w})$ and effective manipulation variance $\sigma_q^2/c(\mathbf{w})^2$. The partial equilibrium characterization, the seller's profit function, and the unconstrained target of $-a_s/a_r$, $M^* = \rho/\sigma_s$ therefore carry over from the baseline. The formal statement is given in Appendix B, and the derivation in the Internet Appendix.

The seller's problem differs from the baseline in that the control variable is the weight vector $\mathbf{w}$ rather than the scalar noise level σ_u. For each $\lambda \in (0, 1]$, define the composite noise index

$$\mathcal{N}\left(\mathbf{w}; \lambda, -\frac{a_s}{a_r}\right) \equiv \sigma_u^2(\mathbf{w}) + a_r\left(\lambda, -\frac{a_s}{a_r}\right)^2 \frac{\sigma_q^2}{c(\mathbf{w})^2}, \tag{29}$$

where $a_r\left(\lambda, -\frac{a_s}{a_r}\right)$ is the equilibrium price to score sensitivity. The second term in (29) already makes the first message of this section clear: because manipulation adds to the noise in the score, the most informative score must account for how manipulable each signal is, not only for its intrinsic precision. Let $M_{\text{sd}}(\lambda)$ denote the multidimensional analogue of $M(\lambda)$ in the baseline: the lowest ratio $-a_s/a_r$ attainable over feasible weight vectors $\mathbf{w}$ at a given participation level λ. A lower ratio means that the price is more informative about the score. Its formal definition is given in Appendix B, where we also note that $M_{\text{sd}}(\lambda)$ exists, is unique, and is decreasing in λ.

For a given λ, denote the most informative weight vector by

$$\mathbf{w}_{\max}(\lambda) \in \arg\min_{\mathbf{w}\in\mathcal{W}} \mathcal{N}(\mathbf{w}; \lambda, M_{\text{sd}}(\lambda))\,. \tag{30}$$

By (30), a signal that is highly precise but easy to manipulate may receive a lower weight in $\mathbf{w}_{\max}$ than pure forecasting considerations would dictate. This is Goodhart's law at work: the firm

manager's manipulation of signal i is proportional to its weight w_i, so placing more weight on a signal makes it a worse indicator of $\tilde{g}$. Importantly, the relative contribution of manipulation to the composite noise is endogenous. It depends on the price sensitivity to the score a_r, which is itself an equilibrium object determined by the investor composition, the share of fundamental investors θ, and investors' risk aversion A. Parameters that are entirely unrelated to the statistical properties of the signals or their manipulability therefore have a direct effect on the optimal score weights. This channel is absent from standard information design problems and arises here because the score simultaneously serves as a target for firm manipulation and as a signal for investor learning.

The following theorem characterizes the equilibrium. Its two cases mirror the unconstrained and constrained cases of the baseline theorem (Theorem 1), with the key distinction that the seller now chooses a weight vector $\mathbf{w}$ in addition to the score fee; Appendix B gives the complete statement.

Theorem 2. *Let $\lambda^\dagger \equiv \min\{\bar{\lambda}, 1\}$, where $\bar{\lambda}$ is defined in Theorem 1.*

1. *If $M_{\text{sd}}(\lambda^\dagger) \leqslant M^*$, the equilibrium is unconstrained. The seller chooses $\lambda^* = \lambda^\dagger$, and any weight vector $\mathbf{w}^* \in \mathcal{W}$ that implements the unconstrained target M^* is optimal. The fraction of score buyers, the score fee, the score variance, and score and price informativeness attain their unconstrained target values.*

2. *Otherwise, the equilibrium is constrained. The seller chooses λ^* to maximize the baseline objective (15) with $M(\lambda)$ replaced by $M_{\text{sd}}(\lambda)$, and the most informative weights, $\mathbf{w}^* = \mathbf{w}_{\max}(\lambda^*)$.*

The economic content of Theorem 2 parallels the baseline. In the unconstrained equilibrium, the seller can reach the unconstrained target M^* and retains slack in the weight choice: any weight vector that generates the target total noise is equally profitable, so the optimal weight vector is generically not unique, and the fraction of score buyers, score fee, score variance, and score and price informativeness attain their unconstrained target values. In the constrained equilibrium, the target is infeasible, so the seller has a strict incentive to minimize the composite noise index $\mathcal{N}$ by choosing $\mathbf{w}_{\max}(\lambda^*)$; this pins down the weights, and manipulation directly affects the score and the price. As with verification in Section 4.3, the additional design tool never raises score or price informativeness above their unconstrained levels: in the unconstrained regime the seller does not

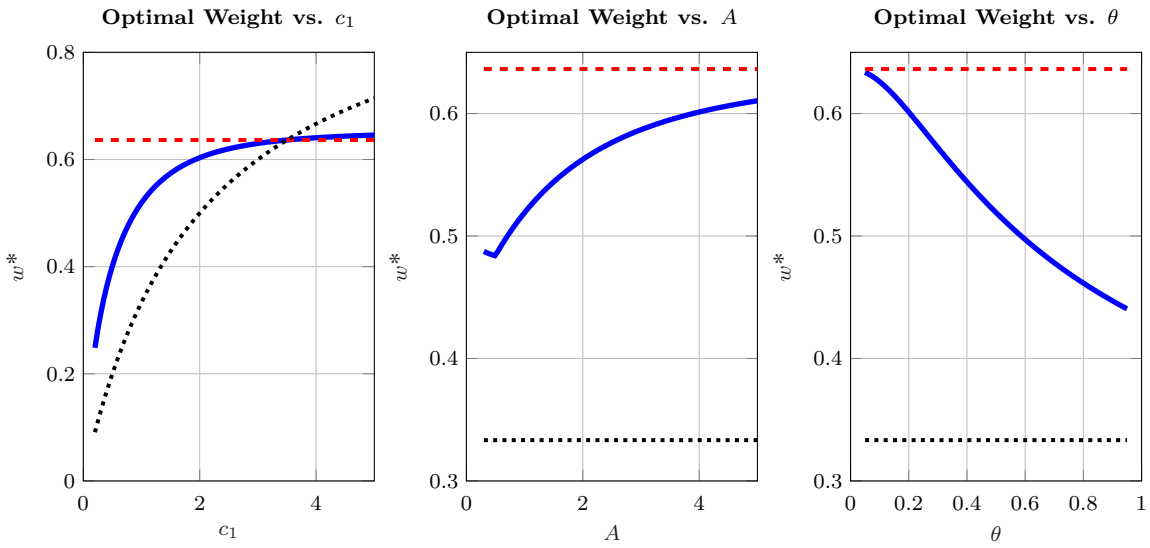


Figure 9: **Optimal Score Weight in the Two Signal Case.** The figure plots the optimal weight w^* on signal 1 in the constrained equilibrium of Theorem 2 as a function of c_1 (left panel), A (middle panel), and θ (right panel). In each panel, the solid blue line is the optimal weight w^*, the dashed red line is the intrinsic precision benchmark w_u, and the dotted black line is the manipulation robust benchmark w_c. Baseline parameter values are reported in Appendix B.1. In each panel, the parameter on the horizontal axis varies while all others are held at their baseline values.

need it, and in the constrained regime even the most informative weights fall short of the target.

Figure 9 illustrates this trade-off with two signals (see Appendix B.1 for the analysis). It plots the optimal weight w^* on signal 1 against two benchmarks: the intrinsic precision benchmark w_u, which minimizes intrinsic noise alone, and the manipulation-robust benchmark w_c, which minimizes manipulation alone. The optimal weight lies between the two (Proposition A.2) and moves toward w_c as the equilibrium intensity of manipulation $a_r^*\sigma_q$ rises. In the figure's parameterization, a_r^* falls with risk aversion A and rises with the mass of fundamental investors θ, so w^* moves toward w_u as A rises and toward w_c as θ rises, even though neither parameter affects the signals' precision or manipulability.

These results have two practical implications. First, a scoring methodology should not be judged by statistical fit alone: weights fitted to the inputs' past predictive power can overweight inputs that become targets for manipulation once the score is in use. Second, disagreement among rating providers need not reflect differences in data or skill, and the model points to two sources. When the equilibrium is unconstrained, providers that use the same signals can rationally choose different

weights, because many weight vectors are equally profitable. This source is more likely to matter where manipulation incentives are less uncertain, and it may contribute to the divergence of credit ratings across agencies (Cantor and Packer, 1997). When the equilibrium is constrained, each provider chooses the most informative weights, but these depend on how manipulable each input is and on the equilibrium intensity of manipulation, $a_r\sigma_q$, not only on the inputs' statistical precision. Providers that agree on the precision of the inputs but assess their manipulability differently therefore choose different weights, and this disagreement grows with manipulation uncertainty. This effect can contribute to the well-documented divergence of ESG ratings (Chatterji et al., 2016; Berg et al., 2022), where uncertainty about firms' greenwashing incentives is likely high.

These results also connect to the policy discussion in Section 4.3. In the unconstrained regime, the seller deliberately chooses weights that are less informative than she could achieve. Raising informativeness in this regime therefore requires limiting the seller's discretion over the design itself, for example through minimum precision standards for scores, and only to the extent that she cannot offset such standards in other ways. In the constrained regime, by contrast, the seller already chooses the most informative weights, so design mandates add little. Outside enforcement that makes specific inputs harder to manipulate is then more useful: it reduces manipulation noise directly, and it allows the seller to place more weight on precise inputs that she would otherwise discount for fear of manipulation.

5 Conclusion

This paper studies a market for manipulable information in which firm managers, investors, and a monopolistic information seller interact strategically. Our framework shows that the effects of information manipulation depend not only on the firm manager's incentive to distort signals, but also on the equilibrium response of investors and the information seller. A central insight is that manipulation affects informativeness only once the intermediary can no longer absorb manipulation-induced noise through its own score design. This gives rise to two distinct regimes: an unconstrained equilibrium, in which manipulation leaves score and price informativeness unchanged, and a constrained equilibrium, in which manipulation reduces the informativeness of both scores

and prices.

The model delivers several implications. First, stronger demand for characteristic-based investing can become self-limiting, weakening the alignment between firms' cost of capital and the targeted characteristic. Second, non-fundamental investors affect market efficiency through their demand elasticity, which shapes firm managers' incentives to manipulate. Third, when investment demand is tied directly to scores and manipulation uncertainty is high, sufficiently strong mandates can backfire by inducing score inflation and reducing price informativeness. This weaker alignment between market valuations and the underlying characteristic suggests that mandates may become less effective at directing capital toward the intended firms through their financing costs. Fourth, when firm managers can choose both real investment and signal manipulation, cheaper manipulation can crowd out real investment. Fifth, even when the information seller can invest in verification that raises the cost of manipulation, score and price informativeness cannot exceed their unconstrained levels: the ceiling reflects the seller's incentives, not her toolkit. Finally, when the score aggregates several manipulable signals, the most informative weights balance signal precision against manipulability and depend on investor composition and risk aversion, not only on the signals' statistical properties; yet a profit-maximizing seller need not choose them, and when she does not, many weight vectors are equally profitable, so providers using the same inputs can produce different scores. Overall, the paper highlights how the design and use of information in financial markets can endogenously distort both prices and real outcomes.

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Appendix

A A Microfoundation for Mandate Demand

This appendix derives the demand function in Eq. (22) of subsection 4.2. Following Pavlova and Sikorskaya (2023), we consider fund managers whose compensation depends on both absolute performance and performance relative to a benchmark. We adapt their mechanism by allowing the benchmark to depend on the reported score and imposing that mandate managers do not learn about payoffs. Proposition A.1 shows how the demand coefficients are determined by the compensation contract and how the contract is recovered from them.

There is a continuum of identical mandate managers with total mass $1-\theta$, each managing wealth W_0 on behalf of a non-fundamental investor. Each manager is infinitesimal and takes the stock price p and reported score g_r as given. A manager chooses ϕ shares of the stock and invests $W_0 - p\phi$ in the riskless bond, generating terminal fund wealth $W_0 + \phi(\tilde{y} - p)$. Borrowing and short sales are unrestricted. The fixed fee for access to the score is absorbed into initial wealth.

Mandate managers have no private information about $\tilde{y}$ and, by assumption, do not learn from either the score or the price. They evaluate portfolios using the unconditional payoff distribution $\tilde{y} \sim N(\bar{y}, \sigma_y^2)$, keeping its mean and variance fixed regardless of the realized (p, g_r). They use the price to calculate the cost of a position and the score to implement the benchmark. This distinguishes them from fundamental investors, who update their payoff beliefs using the price and, when purchased, the score. The absence of learning is an assumption about the managers' evaluation rule, rather than a consequence of the compensation contract. Let $\mathbb{E}_M$ denote expectation under this fixed distribution, treating (p, g_r) as given.

The prescribed benchmark holds

$$\omega(g_r) = \omega_0 + \omega_g g_r, \qquad \omega_g \geqslant 0, \tag{A.1}$$

shares of the stock and $W_0 - p\omega(g_r)$ in the riskless bond. As in Pavlova and Sikorskaya (2023), the benchmark is specified in shares, so its stock holding is fixed as the price varies, holding the score

constant. The coefficients ω_0 and ω_g are exogenous and do not depend on score precision. The fund's gain relative to investing in the riskless bond is $\phi(\tilde{y} - p)$, and the corresponding benchmark gain is $\omega(g_r)(\tilde{y} - p)$. The manager receives compensation

$$C(\phi) = c_0 + a\phi(\tilde{y} - p) + b[\phi - \omega(g_r)](\tilde{y} - p), \qquad a > 0, \quad b \geqslant 0, \tag{A.2}$$

where a rewards absolute performance, b rewards performance relative to the benchmark, and c_0 is a fixed payment.

The manager takes the contract as given, has no other risky income, and has CARA utility over compensation with risk aversion $\gamma > 0$. The portfolio problem is

$$\max_{\phi \in \mathbb{R}} \ \mathbb{E}_M\Big[-e^{-\gamma C(\phi)}\Big].$$

Under the fixed normal payoff distribution, this is equivalent to maximizing the certainty equivalent

$$c_0 + [(a+b)\phi - b\omega(g_r)](\bar{y} - p) - \frac{\gamma\sigma_y^2}{2}[(a+b)\phi - b\omega(g_r)]^2.$$

The objective is strictly concave, with second derivative $-\gamma\sigma_y^2(a+b)^2 < 0$. Its first-order condition, after dividing by $a + b$, is

$$\bar{y} - p - \gamma\sigma_y^2[(a+b)\phi - b\omega(g_r)] = 0.$$

Hence the unique optimal holding is

$$\phi^*(p, g_r) = \frac{\bar{y} - p}{\gamma(a+b)\sigma_y^2} + \frac{b}{a+b}(\omega_0 + \omega_g g_r). \tag{A.3}$$

Since each manager demands $\phi^*(p, g_r)$ shares, aggregate mandate demand is $(1 - \theta)\phi^*(p, g_r)$.

Equation (A.3) expresses the demand in terms of the manager's risk aversion γ, the compensation contract (a, b), and the benchmark (ω_0, ω_g). Taking γ and the benchmark as given, the following proposition states the relation between the contract and the demand function in our model $D_0 - D_1 p + D_2 g_r$ in both directions.

Proposition A.1. *Fix $\gamma > 0$ and a benchmark (ω_0, ω_g) with $\omega_g > 0$.*

1. *Given a contract (a, b) with $a > 0$ and $b \geqslant 0$, the manager's demand is $D_0 - D_1 p + D_2 g_r$ with*

$$D_1 = \frac{1}{\gamma(a+b)\sigma_y^2}, \qquad D_2 = \frac{b}{a+b}\,\omega_g, \qquad D_0 = \frac{\bar{y}}{\gamma(a+b)\sigma_y^2} + \frac{b}{a+b}\,\omega_0.$$

2. *Given (D_1, D_2) with $D_1 > 0$ and $0 \leqslant D_2 < \omega_g$, the contract that generates them is unique:*

$$a = \frac{1 - D_2/\omega_g}{\gamma D_1 \sigma_y^2}, \qquad b = \frac{D_2/\omega_g}{\gamma D_1 \sigma_y^2}. \tag{A.4}$$

The intercept is then implied by the slopes, $D_0 = D_1\bar{y} + (\omega_0/\omega_g)D_2$.

Proof. Part 1 is (A.3). For Part 2, the expressions for D_1 and D_2 in Part 1 give $a + b = 1/(\gamma D_1 \sigma_y^2)$ and $b/(a+b) = D_2/\omega_g$. Solving for a and b yields (A.4), with $b \geqslant 0$ and $a > 0$ because $0 \leqslant D_2 < \omega_g$. Substituting $b/(a+b) = D_2/\omega_g$ into the expression for D_0 gives the implied intercept. □

The intercept D_0 is thus implied by the slopes rather than chosen freely. This restriction is innocuous: D_0 enters the equilibrium only through the mean price in (A.32) and plays no role in any of our results, so we do not discuss it further.

Remark A.1. *The baseline demand $D_0 - D_1 p$ of Section 2 is the special case of a score-independent benchmark, $\omega_g = 0$, which gives $D_2 = 0$ and $D_0 = D_1\bar{y} + \frac{b}{a+b}\omega_0$. It includes pure absolute-performance contracts, $b = 0$, for which $D_0 = D_1\bar{y}$. The microfoundation therefore covers both the baseline and the mandate extension.*

B Multidimensional Score Design: Formal Details

This appendix collects the formal objects behind Theorem 2 in subsection 4.4. As in (29), the ratio argument of a_r and $\mathcal{N}$ is $-a_s/a_r$. Evaluations at $M_{\mathrm{sd}}(\lambda)$ or M^* use the lowest attainable ratio or the unconstrained target, respectively. For each $\lambda \in (0, 1]$, define $M_{\mathrm{sd}}(\lambda)$ as the unique solution of

$$\rho^2 + \frac{\rho\,\theta\lambda}{A\sigma_y}\,M_{\mathrm{sd}}(\lambda) = 1 + \min_{\mathbf{w}\in\mathcal{W}} \mathcal{N}(\mathbf{w}; \lambda, M_{\mathrm{sd}}(\lambda))\,. \tag{A.5}$$

where $\mathcal{N}$ is the composite noise index defined in subsection 4.4. The proof of Theorem 2 in the Internet Appendix shows that $M_{\mathrm{sd}}(\lambda)$ exists, is unique, and is decreasing in λ.

We assume that for every $\lambda \in (0,1]$, whenever $M_{\mathrm{sd}}(\lambda) \leqslant M^*$, there exists a weight vector $\mathbf{w} \in \mathcal{W}$ such that

$$1 + \mathcal{N}(\mathbf{w}; \lambda, M^*) = \rho^2 \left(1 + \frac{\theta\lambda}{A\sigma_s\sigma_y}\right).$$

That is, whenever the most informative feasible design is at least as informative as the unconstrained target, $M_{\mathrm{sd}}(\lambda) \leqslant M^*$, the seller can choose weights to implement M^* exactly.

The following theorem gives the complete statement of Theorem 2.

Theorem 3. *In the multidimensional score design extension, the seller's optimal fee Φ^* and weight vector $\mathbf{w}^*$ are characterized as follows.*

1. *If $\bar{\lambda} \leqslant 1$ and $M_{\mathrm{sd}}(\bar{\lambda}) \leqslant M^*$, then the equilibrium is unconstrained. The seller chooses $\lambda^* = \bar{\lambda}$, sets the corresponding fee Φ^*, and any $\mathbf{w}^* \in \mathcal{W}$ satisfying*

$$1 + \mathcal{N}(\mathbf{w}^*; \bar{\lambda}, M^*) = \rho^2 \left(1 + \frac{\theta\bar{\lambda}}{A\sigma_s\sigma_y}\right) \tag{A.6}$$

   is optimal.

2. *If $\bar{\lambda} > 1$ and $M_{\mathrm{sd}}(1) \leqslant M^*$, then the equilibrium is again unconstrained. The seller chooses $\lambda^* = 1$, sets the corresponding fee Φ^*, and any $\mathbf{w}^* \in \mathcal{W}$ satisfying*

$$1 + \mathcal{N}(\mathbf{w}^*; 1, M^*) = \rho^2 \left(1 + \frac{\theta}{A\sigma_s\sigma_y}\right) \tag{A.7}$$

   is optimal.

3. *Otherwise, the equilibrium is constrained. The seller chooses*

$$\lambda^* \in \arg\max_{\lambda \in [0,1]} \lambda \ln\left(1 + \frac{A\sigma_s^2\sigma_y^2}{\theta\lambda\left(\frac{\rho\sigma_y}{M_{\mathrm{sd}}(\lambda)} + \frac{\theta\lambda}{A} + \frac{\sigma_s^2\sigma_y}{\rho} M_{\mathrm{sd}}(\lambda)\right)}\right), \tag{A.8}$$

   where the objective is defined as zero at $\lambda = 0$. This is the same objective as in (15), with

$M(\lambda)$ replaced by $M_{\mathrm{sd}}(\lambda)$. An optimal weight vector is given by

$$\mathbf{w}^* = \mathbf{w}_{\max}(\lambda^*). \tag{A.9}$$

B.1 Two-signal case

To obtain sharper comparative statics, we specialize to $N = 2$ and focus on the constrained equilibrium. Write $w_1 = w$ and $w_2 = 1 - w$ for $w \in [0, 1]$. Given the equilibrium values λ^* and $M_{\mathrm{sd}}(\lambda^*)$, the seller's weight choice reduces to

$$\min_{w\in[0,1]} L(w) \equiv V_u(w) + a_r^{*2}\sigma_q^2\, m(w)^2, \tag{A.10}$$

where

$$V_u(w) = w^2(\Sigma_u)_{11} + 2w(1-w)(\Sigma_u)_{12} + (1-w)^2(\Sigma_u)_{22}, \tag{A.11}$$

$$m(w) = \frac{w^2}{c_1} + \frac{(1-w)^2}{c_2}, \tag{A.12}$$

and $a_r^* = a_r(\lambda^*, M_{\mathrm{sd}}(\lambda^*))$ is the equilibrium price sensitivity to the score. The objective $L(w)$ decomposes the seller's problem into two channels: minimizing intrinsic noise (through V_u) and minimizing the firm manager's equilibrium manipulation (through m).

Two natural benchmarks emerge. The intrinsic precision benchmark w_u minimizes $V_u(w)$ alone:

$$w_u = \frac{(\Sigma_u)_{22} - (\Sigma_u)_{12}}{(\Sigma_u)_{11} + (\Sigma_u)_{22} - 2(\Sigma_u)_{12}}. \tag{A.13}$$

The manipulation robust benchmark w_c minimizes $m(w)$ alone:

$$w_c = \frac{c_1}{c_1 + c_2}. \tag{A.14}$$

The benchmark w_c places more weight on the signal that is more costly to manipulate, since the firm manager allocates less manipulation to that signal in equilibrium.

Proposition A.2. *Assume $0 < w_u < 1$. Then L is strictly convex and the optimal weight w^* is unique, with*

$$w^* \in \big[\min\{w_u, w_c\},\ \max\{w_u, w_c\}\big]. \tag{A.15}$$

subsection 4.4 discusses how w^* varies with the equilibrium intensity of manipulation $a_r^*\sigma_q$ and illustrates it in Figure 9. The interval result (A.15) follows because deviating beyond either benchmark would increase both components of L simultaneously.

Figure 9 uses the baseline parameter values $\sigma_y = 2$, $\sigma_s = 0.5$, $\sigma_g = 1$, $\sigma_q = 1.5$, $\rho = 0.8$, $A = 1$, $\theta = 0.5$, $D_1 = 0.5$, $c_1 = 1$, $c_2 = 2$, $(\Sigma_u)_{11} = 0.5$, $(\Sigma_u)_{22} = 0.8$, and $(\Sigma_u)_{12} = 0.1$. Here $(\Sigma_u)_{ij} = \mathrm{Cov}(u_i, u_j)$ denotes the covariance of the intrinsic signal errors.

C Proofs

C.1 Proof of Proposition 1

Fix $\sigma_u \geqslant \underline{\sigma}$ and $\lambda \in (0, 1]$. We derive the manager's strategy and investors' demands, then establish existence and uniqueness of the price coefficients.

1. Manipulation and posterior beliefs. Conjecture a linear price $p = a_0 + a_r g_r + a_s \tilde{s}$ with $a_r > 0$ and $a_s < 0$. The manager's objective in (4) is strictly concave in δ, with first-order condition $qa_r - c\delta = 0$. Thus $\delta^* = a_r q/c$ and, by (5), $g_r = \tilde{g} + u + a_r q/c$.

Let $\mathscr{F}_I$ and $\mathscr{F}_U$ be generated by (g_r, p) and p, respectively, and write $\bar{g}_r = a_r\bar{q}/c$ and $\bar{p} = \mathbb{E}[p]$. Define

$$r_1 \equiv \frac{1}{1 + \sigma_u^2 + a_r^2\sigma_q^2/c^2}, \qquad r_2 \equiv \frac{1}{1 + \sigma_u^2 + a_r^2\sigma_q^2/c^2 + (a_s/a_r)^2\sigma_s^2}. \tag{A.16}$$

Given g_r, the price reveals only supply. Gaussian updating therefore gives

$$\mathbb{E}[\tilde{y} \mid \mathscr{F}_I] = \bar{y} + \rho\sigma_y r_1(g_r - \bar{g}_r), \qquad \mathrm{Var}[\tilde{y} \mid \mathscr{F}_I] = V_1, \tag{A.17}$$

$$\mathbb{E}[\tilde{y} \mid \mathscr{F}_U] = \bar{y} + \rho\sigma_y r_2\frac{p - \bar{p}}{a_r}, \qquad \mathrm{Var}[\tilde{y} \mid \mathscr{F}_U] = V_2, \tag{A.18}$$

where $V_i \equiv \sigma_y^2(1-\rho^2 r_i) > 0$ for $i = 1, 2$, since $\sigma_u \geqslant \underline{\sigma} > 0$.

2. Market clearing and price loadings. CARA–normal portfolio choice gives $\phi_i = (\mathbb{E}[\tilde{y} \mid \mathscr{F}_i] - p)/(A \operatorname{Var}[\tilde{y} \mid \mathscr{F}_i])$ for $i \in \{I, U\}$. Market clearing requires

$$\theta\{\lambda\phi_I + (1-\lambda)\phi_U\} + (1-\theta)(D_0 - D_1 p) = \tilde{s}. \tag{A.19}$$

Substituting the posterior moments and multiplying by A yields

$$\begin{aligned} &\theta\lambda\frac{\bar{y} + \rho\sigma_y r_1(g_r - \bar{g}_r) - p}{V_1} + \theta(1-\lambda)\frac{\bar{y} + \rho\sigma_y r_2(p - \bar{p})/a_r - p}{V_2} \\ &\quad + A(1-\theta)(D_0 - D_1 p) = A\tilde{s}. \end{aligned} \tag{A.20}$$

Taking differentials gives

$$\left[\theta\left(\frac{\lambda}{V_1} + \frac{(1-\lambda)(1-\rho\sigma_y r_2/a_r)}{V_2}\right) + A(1-\theta)D_1\right] dp = \theta\frac{\lambda\rho\sigma_y r_1}{V_1} dg_r - A\, d\tilde{s}. \tag{A.21}$$

Matching coefficients with $dp = a_r dg_r + a_s d\tilde{s}$ gives

$$\left[\theta\left(\frac{\lambda}{V_1} + \frac{(1-\lambda)(1-\rho\sigma_y r_2/a_r)}{V_2}\right) + A(1-\theta)D_1\right] a_s = -A, \tag{A.22}$$

$$\left[\theta\left(\frac{\lambda}{V_1} + \frac{(1-\lambda)(1-\rho\sigma_y r_2/a_r)}{V_2}\right) + A(1-\theta)D_1\right] a_r = \theta\frac{\lambda\rho\sigma_y r_1}{V_1}. \tag{A.23}$$

Dividing these equations and using $V_1 = \sigma_y^2(1-\rho^2 r_1)$ yields

$$\frac{a_s}{a_r} = -\frac{AV_1}{\theta\lambda\rho\sigma_y r_1} < 0, \tag{A.24}$$

$$\frac{1}{r_1} = \rho^2 + \frac{\rho\theta\lambda}{A\sigma_y} z, \tag{A.25}$$

$$\frac{1}{r_2} - \rho^2 = \frac{1}{r_1} - \rho^2 + z^2\sigma_s^2 = \frac{\rho\theta\lambda}{A\sigma_y} z + z^2\sigma_s^2, \tag{A.26}$$

where $z \equiv -a_s/a_r > 0$. Rearranging (A.23) gives

$$\left[\theta\left(\frac{\lambda}{V_1} + \frac{1-\lambda}{V_2}\right) + A(1-\theta)D_1\right] a_r = \theta\rho\sigma_y\left(\frac{\lambda r_1}{V_1} + \frac{(1-\lambda)r_2}{V_2}\right). \tag{A.27}$$

Both the coefficient and the right-hand side are positive, implying $a_r > 0$ and hence $a_s < 0$. Substituting $V_i = \sigma_y^2(1 - \rho^2 r_i)$ and (A.25)–(A.26) gives

$$\frac{1}{a_r} = \frac{\rho}{\sigma_y} + \left(\frac{\theta}{A\sigma_y^2} + (1-\theta)D_1 \right) z \frac{A\sigma_s^2\sigma_y^2 z + \rho\sigma_y\theta\lambda}{A\sigma_s^2\sigma_y^2 z + \rho\sigma_y\theta}. \tag{A.28}$$

Thus a_r is a positive function of z, and $a_s = -z a_r$.

3. Existence, uniqueness, and the intercept. The fraction in (A.28) is positive and nondecreasing in z because $0 < \lambda \leqslant 1$. Its product with z is therefore strictly increasing, so $a_r(z)$ is strictly decreasing. The definition of r_1 gives

$$\frac{1}{r_1} = 1 + \sigma_u^2 + a_r^2 \frac{\sigma_q^2}{c^2}. \tag{A.29}$$

Combining this with (A.25) yields

$$\rho^2 + \frac{\rho\theta\lambda}{A\sigma_y} z = 1 + \sigma_u^2 + a_r(z)^2 \frac{\sigma_q^2}{c^2}. \tag{A.30}$$

The left-hand side is strictly increasing in z and diverges, while the right-hand side is continuous, bounded, and weakly decreasing. At $z = 0$, the left-hand side is strictly smaller because $\rho^2 \leqslant 1 < 1 + \sigma_u^2$. Hence there is a unique solution $z > 0$, which uniquely determines $a_r > 0$ and $a_s = -z a_r < 0$.

Taking expectations in (A.20) determines the mean price:

$$\bar{p} = \frac{\theta \left(\frac{\lambda}{V_1} + \frac{1-\lambda}{V_2} \right) \bar{y} + A(1-\theta)D_0 - A\bar{s}}{\theta \left(\frac{\lambda}{V_1} + \frac{1-\lambda}{V_2} \right) + A(1-\theta)D_1}. \tag{A.31}$$

Since $\bar{p} = a_0 + a_r \bar{g}_r + a_s \bar{s}$ and $\bar{g}_r = a_r \bar{q}/c$, the unique intercept is

$$a_0 = \bar{p} - \left(a_r^2 \frac{\bar{q}}{c} + a_s \bar{s} \right). \tag{A.32}$$

The resulting price and demands clear the market, and $\delta^* = a_r q/c$ is the manager's unique optimum. This establishes the stated equilibrium and uniquely determines its price coefficients. □

C.2 Proof of Theorem 1

Use $z = -a_s/a_r$, r_1, r_2, and V_1, V_2 from the proof of Proposition 1.

1. Revenue and feasibility. By Lemma 1 and (A.25)–(A.26), revenue is $\theta\Pi(\lambda, z)/(2A)$, where

$$\Pi(\lambda, z) \equiv \lambda \ln \frac{V_2}{V_1} = \lambda \ln \frac{1-\rho^2 r_2}{1-\rho^2 r_1}, \tag{A.33}$$

$$\frac{V_2}{V_1} = 1 + \frac{A\sigma_s^2\sigma_y^2}{\theta\lambda\left(\rho\sigma_y/z + \theta\lambda/A + \sigma_s^2\sigma_y z/\rho\right)}. \tag{A.34}$$

For fixed $\lambda > 0$, $a_r(z)$ strictly decreases by (A.28). Thus (A.30) makes σ_u strictly increasing in z, with feasible ratios exactly $[M(\lambda), \infty)$. The seller's problem at positive participation is therefore

$$\max_{0<\lambda\leqslant 1,\ z\geqslant M(\lambda)} \Pi(\lambda, z). \tag{A.35}$$

The same equations imply that M is continuous and strictly decreasing: at fixed z, increasing λ raises the left-hand side of (A.30) and lowers a_r on the right-hand side. Also, (11) gives $M(\lambda) \geqslant A\sigma_y(1+\underline{\sigma}^2-\rho^2)/(\rho\theta\lambda) \to \infty$ as $\lambda \downarrow 0$.

2. The unconstrained optimum. For fixed $\lambda > 0$, maximizing (A.34) amounts to minimizing $\rho\sigma_y/z + \sigma_s^2\sigma_y z/\rho$. Its unique minimizer and the resulting objective are

$$z = M^* = \frac{\rho}{\sigma_s}, \tag{A.36}$$

$$\Pi(\lambda, M^*) = \lambda \ln\left(1 + \frac{A^2\sigma_s^2\sigma_y^2}{\theta\lambda(\theta\lambda + 2A\sigma_s\sigma_y)}\right). \tag{A.37}$$

With $k = \theta\lambda/(A\sigma_s\sigma_y)$, the latter is proportional to $f(k) = k\ln(1+1/[k(k+2)])$. Differentiation gives $f'(k) = \ln(1+1/[k(k+2)]) - 2/[(k+1)(k+2)]$, while $f''(k)$ has the sign of k^2-2. Thus f' decreases from infinity on $(0, \sqrt{2})$ and then increases to zero from below. Its unique zero, $\lambda_0 \approx 0.65146$, uniquely maximizes f. This yields (12) and the relaxed optimum $(\min\{\overline{\lambda}, 1\}, M^*)$.

Since $\Pi(\lambda, z)$ increases up to M^* and decreases thereafter, the optimal feasible ratio is $\max\{M(\lambda), M^*\}$. The resulting value is continuous for $\lambda > 0$. For feasible choices, $0 < \Pi(\lambda, z) \leqslant -\lambda\ln[1-\rho^2/(1+\underline{\sigma}^2)]$,

so it extends continuously to zero at $\lambda = 0$. A positive maximum therefore exists on $[0, 1]$.

3. The three cases. Unconstrained equilibrium (partial and full participation). Under their respective strict inequalities, the relaxed optimum is feasible with $\sigma_u^* > \underline{\sigma}$ and $z = M^*$. Participation is $\lambda^* = \overline{\lambda}$ with partial participation ($\overline{\lambda} < 1$) and $\lambda^* = 1$ with full participation ($\overline{\lambda} \geqslant 1$). Substituting these choices into (A.25), with $\mathrm{Var}(g_r) = 1/r_1$, gives (13) and (14), respectively. Equation (A.28) gives the corresponding loadings:

$$\frac{1}{a_r^*} = \frac{\rho}{\sigma_y} + \left(\frac{\theta}{A\sigma_y^2} + (1-\theta)D_1\right)\frac{\rho}{\sigma_s}\frac{A\sigma_s\sigma_y + \theta\overline{\lambda}}{A\sigma_s\sigma_y + \theta}, \tag{A.38}$$

$$\frac{1}{a_r^{**}} = \frac{\rho}{\sigma_y} + \left(\frac{\theta}{A\sigma_y^2} + (1-\theta)D_1\right)\frac{\rho}{\sigma_s}. \tag{A.39}$$

Constrained equilibrium. Otherwise, $M(\min\{\overline{\lambda}, 1\}) \geqslant M^*$. If an optimum had slack noise, local optimality in z would require $z = M^*$. Continuity of M would permit local participation changes at fixed z, so strict unimodality would require $\lambda = \min\{\overline{\lambda}, 1\}$, with one-sided changes at the cap. This contradicts slack because that target is infeasible or on the noise floor. Hence every optimum has $\sigma_u^* = \underline{\sigma}$ and $z = M(\lambda^*)$. Substituting into (A.35) gives (15), with the objective extended by zero at $\lambda = 0$.

In each case, (9) gives the highest fee sustaining λ^*, completing the characterization. □

C.3 Proof of Proposition 2

Use superscript 0 for the benchmark $q \equiv 0$ and evaluate unmarked quantities at the equilibrium with manipulation.

1. Participation and score variance. In (11), setting $\sigma_q = 0$ weakly lowers $M(\lambda)$ at each fixed $\lambda > 0$, while $\overline{\lambda}$ and $M^* = \rho/\sigma_s$ are unchanged. Thus the strict condition for the unconstrained equilibrium, with partial or full participation, continues to hold in the benchmark. Applying Theorem 1 to both

economies gives $z = M^*$ and

$$\lambda^{0*} = \lambda^* = \min\{\overline{\lambda}, 1\}, \tag{A.40}$$

$$\mathrm{Var}(g_r) = \rho^2 \left(1 + \frac{\theta\lambda^*}{A\sigma_s\sigma_y}\right), \tag{A.41}$$

$$\mathrm{Var}(g_r^0) = \mathrm{Var}(g_r). \tag{A.42}$$

By (6) and (5), the scores and noise choices therefore satisfy

$$g_r = \tilde{g} + u + a_r q/c, \tag{A.43}$$

$$g_r^0 = \tilde{g} + u^0, \tag{A.44}$$

$$(\sigma_u^{0*})^2 = (\sigma_u^*)^2 + a_r^2\sigma_q^2/c^2. \tag{A.45}$$

The benchmark replaces manipulation noise with seller noise and remains interior, since $\sigma_u^{0*} \geqslant \sigma_u^* > \underline{\sigma}$.

2. Price coefficients and score fee. The common (λ^*, z) gives the same a_r and a_s by (A.28) and $a_s = -za_r$. The common score variance and loadings also give the same r_1, r_2, V_1, V_2 in (A.16)–(A.18). Hence (9) implies

$$\Phi(\lambda) = \frac{1}{2A} \ln \frac{V_2}{V_1}, \tag{A.46}$$

$$\Phi^{0*} = \Phi^*. \tag{A.47}$$

3. Informativeness and distributions. Equations (A.31) and (A.32) give a common mean price $\bar{p}$ and the centered representation $p - \bar{p} = a_r(g_r - \bar{g}_r) + a_s(\tilde{s} - \bar{s})$, where $\bar{g}_r = a_r\bar{q}/c$ and the benchmark has $\bar{g}_r^0 = 0$. The noises $u + a_r(q - \bar{q})/c$ and u^0 are mean-zero Gaussian, independent of $(\tilde{y}, \tilde{g}, \tilde{s})$, and have the same variance by (A.45). Thus the centered scores have the same joint distribution with $(\tilde{y}, \tilde{g}, \tilde{s})$, as do the prices through their common centered representation. Since centering leaves

correlations unchanged,

$$\operatorname{corr}(g_r, \tilde{g}) = \operatorname{corr}(g_r^0, \tilde{g}), \tag{A.48}$$

$$\operatorname{corr}(p, \tilde{g}) = \operatorname{corr}(p^0, \tilde{g}), \qquad \operatorname{corr}(p, g_r) = \operatorname{corr}(p^0, g_r^0). \tag{A.49}$$

This also establishes equality of stock-price distributions. The posterior means in (A.17)–(A.18), common conditional variances, and portfolio demands in the proof of Proposition 1 imply the same joint distribution of fundamental investors' holdings. Non-fundamental holdings $D_0 - D_1 p$ have the same joint distribution with them. Together with Steps 1–2, this proves the proposition. □

C.4 Proof of Proposition 3

Write $M(\lambda; \sigma_q)$ to make the dependence on manipulation uncertainty explicit. Let $D(M, \lambda) = 1/a_r$ denote the right-hand side of (A.28) with $z = M$, and set $\kappa = \theta/(A\sigma_y^2) + (1-\theta)D_1$.

1. The binding frontier. Equation (11) is equivalent to

$$F(M, \sigma_q; \lambda) \equiv \rho^2 + \frac{\rho\theta\lambda}{A\sigma_y} M - 1 - \underline{\sigma}^2 - \frac{\sigma_q^2}{c^2 D(M, \lambda)^2} = 0.$$

The proof of Proposition 1 gives a unique positive root for each $\lambda \in (0, 1]$. Since $D_M, D_\lambda > 0$, we have $F_M, F_\lambda > 0$ and $F_{\sigma_q} < 0$ for $\sigma_q > 0$. Thus M is continuous, strictly increasing in σ_q, and strictly decreasing in λ. Also, $M(\lambda; \sigma_q) \to \infty$ as $\sigma_q \to \infty$: otherwise M and D would remain bounded along a sequence, contradicting (11).

2. The constrained-regime threshold. Set $\lambda^\dagger = \min\{\overline{\lambda}, 1\}$ and $M^* = \rho/\sigma_s$. By Theorem 1, the equilibrium is constrained exactly when $M(\lambda^\dagger; \sigma_q) \geqslant M^*$. Step 1 therefore gives the finite threshold

$$\sigma_q^\dagger = \inf\{\sigma_q \geqslant 0 : M(\lambda^\dagger; \sigma_q) \geqslant M^*\}. \tag{A.50}$$

Evaluating F at M^* yields

$$\sigma_q^\dagger = c\rho\left[\frac{1}{\sigma_y} + \frac{\kappa}{\sigma_s}\frac{A\sigma_s\sigma_y + \theta\lambda^\dagger}{A\sigma_s\sigma_y + \theta}\right] \times \sqrt{\max\left\{\rho^2\left(1 + \frac{\theta\lambda^\dagger}{A\sigma_s\sigma_y}\right) - 1 - \underline{\sigma}^2,\, 0\right\}}. \tag{A.51}$$

This includes $\sigma_q^\dagger = 0$ when the noise floor already binds at $\sigma_q = 0$. Continuity and strict monotonicity imply that the equilibrium is constrained if and only if $\sigma_q \geqslant \sigma_q^\dagger$.

3. Full participation uniformly in θ. Differentiating (A.28) gives $0 < MD_M/D \leqslant 2$. At the root, $\sigma_q^2/(c^2D^2) = \rho\theta\lambda M/(A\sigma_y) - (1 + \underline{\sigma}^2 - \rho^2) < \rho\theta\lambda M/(A\sigma_y)$, so

$$F_M \leqslant \frac{5\rho\theta\lambda}{A\sigma_y}, \qquad F_\lambda \geqslant \frac{\rho\theta M}{A\sigma_y}, \qquad -M_\lambda = \frac{F_\lambda}{F_M} \geqslant \frac{M}{5\lambda}.$$

Choose $M_0 = \rho/\sigma_s + 5\rho/(A\sigma_s^2\sigma_y)$, so that $\sigma_s^2\sigma_y M_0/\rho - \rho\sigma_y/M_0 > 5/A$, and define

$$\sigma_q^{\text{full}} = c\left[\frac{\rho}{\sigma_y} + M_0 \max\left\{D_1, \frac{1}{A\sigma_y^2}\right\}\right]\sqrt{\frac{\rho M_0}{A\sigma_y}}.$$

This bound is independent of θ. Since $D(M,\lambda) \leqslant \rho/\sigma_y + \kappa M$ and $\kappa \leqslant \max\{D_1, 1/(A\sigma_y^2)\}$, the root identity above gives $\sigma_q < \sigma_q^{\text{full}}$ whenever $M \leqslant M_0$. Thus $\sigma_q \geqslant \sigma_q^{\text{full}}$ forces $M(\lambda;\sigma_q) > M_0$ for every $\theta, \lambda \in (0,1]$. Hence $M > M^*$ throughout, the equilibrium is constrained, and $\sigma_q^{\text{full}} \geqslant \sigma_q^\dagger$.

Write $H(\lambda) = \rho\sigma_y/M + \theta\lambda/A + \sigma_s^2\sigma_y M/\rho$, with $M = M(\lambda;\sigma_q)$. The derivative bound and choice of M_0 give

$$\lambda H'(\lambda) = \frac{\theta\lambda}{A} + \frac{\lambda M_\lambda}{M}\left(\frac{\sigma_s^2\sigma_y M}{\rho} - \frac{\rho\sigma_y}{M}\right) < \frac{\theta\lambda - 1}{A} \leqslant 0.$$

For fixed $H > 0$, the objective $\Pi(\lambda, M(\lambda)) = \lambda\ln[1 + A\sigma_s^2\sigma_y^2/(\theta\lambda H)]$ is strictly increasing in λ, since $\ln(1+x) > x/(1+x)$ for $x > 0$. Decreasing H raises it further, so the objective of the constrained equilibrium is strictly increasing on $(0,1]$. Zero participation gives zero revenue, whereas positive participation gives positive revenue. Thus $\lambda^* = 1$ uniquely for every $\sigma_q \geqslant \sigma_q^{\text{full}}$ and $\theta \in (0,1]$.

4. Informativeness. In constrained equilibrium, write $z = M(\lambda^*;\sigma_q)$. Equations (A.25)–(A.26)

imply

$$\operatorname{corr}(g_r, \tilde{g}) = \left[\rho^2 + \frac{\rho\theta}{A\sigma_y}\lambda^* z\right]^{-1/2},$$
$$\operatorname{corr}(p, \tilde{g}) = \left[\rho^2 + \frac{\rho\theta}{A\sigma_y}\lambda^* z + \sigma_s^2 z^2\right]^{-1/2}.$$

The Internet Appendix subsection titled "Proof of Part 2 of Proposition 3" establishes that both z and $\lambda^* z$ strictly increase with σ_q, accounting for the seller's endogenous participation choice and transitions at the participation cap. These formulas therefore imply that both correlations strictly decrease with σ_q throughout the constrained regime, proving part 2. □

Internet Appendix for

"Market for Manipulable Information"

Hui Chen[1] and Jian Sun[2]

IA.1 Additional Proofs

IA.1.1 Proof of Corollary 1

Fix $\sigma_u \geqslant \underline{\sigma}$ and $\lambda \in (0, 1]$, holding all other primitives except σ_q fixed. Use $z = -a_s/a_r$ and r_1, r_2 from the proof of Proposition 1.

1. Price sensitivity. As established in Step 3 of that proof, (A.28) defines a positive, strictly decreasing function $a_r(z)$, and (A.30) determines a unique $z > 0$. In the latter equation, the left-hand side strictly increases in z, while the right-hand side weakly decreases in z and strictly increases in σ_q. Raising σ_q therefore increases the unique equilibrium z and lowers a_r.

2. Manipulation variance. By (6) and (A.30), $\text{Var}(\delta^*) = a_r^2\sigma_q^2/c^2 = \rho^2 + \frac{\rho\theta\lambda}{A\sigma_y}z - 1 - \sigma_u^2$. Its coefficient on z is positive, so Step 1 implies that manipulation variance strictly increases with σ_q.

3. Score and price informativeness. The price is $p = a_0 + a_r(g_r - z\tilde{s})$ with $a_r > 0$, and supply is independent of g_r and $\tilde{g}$. Using (A.16), the three correlations are $\text{corr}(g_r, \tilde{g}) = \sqrt{r_1}$, $\text{corr}(p, \tilde{g}) = \sqrt{r_2}$, and $\text{corr}(p, g_r) = \sqrt{r_2/r_1}$. Equations (A.25)–(A.26) show that r_1 and r_2 strictly decrease in z, so the first two correlations decrease with σ_q. For the third, $r_2/r_1 = (1 + \sigma_s^2 z^2 r_1)^{-1}$, where $z^2 r_1 = [\rho^2/z^2 + \rho\theta\lambda/(A\sigma_y z)]^{-1}$ strictly increases in z. Thus the third correlation also strictly decreases with σ_q, completing the proof. □

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IA.1.2 Proof of Lemma 1

Fix $\lambda \in (0, 1]$ and use the information sets $\mathscr{F}_I, \mathscr{F}_U$ and posterior variances V_1, V_2 from the proof of Proposition 1. Write $\mu_I = \mathbb{E}[\tilde{y} \mid \mathscr{F}_I]$ and $\mu_U = \mathbb{E}[\tilde{y} \mid \mathscr{F}_U]$; Gaussian updating makes both posterior variances constant.

1. Conditional utility. Before paying a score fee, the investor's budget is $l = W_0 - \phi p$, so terminal wealth is $W_0 + \phi(\tilde{y} - p)$. Substituting the portfolio demands derived in the proof of Proposition 1 into (2) gives the optimized conditional utilities

$$U_I = -\exp\left[-AW_0 - \frac{(\mu_I - p)^2}{2V_1}\right], \tag{IA.1}$$

$$U_U = -\exp\left[-AW_0 - \frac{(\mu_U - p)^2}{2V_2}\right]. \tag{IA.2}$$

2. The value of the score. The nested Gaussian information sets imply $\mu_I \mid \mathscr{F}_U \sim N(\mu_U, V_2 - V_1)$. Since p is observed under $\mathscr{F}_U$, completing the square gives

$$\mathbb{E}[U_I \mid \mathscr{F}_U] = -e^{-AW_0}\mathbb{E}\left[\exp\left(-\frac{(\mu_I - p)^2}{2V_1}\right)\middle| \mathscr{F}_U\right] \tag{IA.3}$$

$$= \sqrt{\frac{V_1}{V_2}}\, U_U. \tag{IA.4}$$

The factor $\sqrt{V_1/V_2}$ is constant across price realizations. Taking expectations again therefore gives $\mathbb{E}[U_I] = \sqrt{V_1/V_2}\,\mathbb{E}[U_U]$, so the same utility comparison applies when investors decide whether to buy the score, before observing either the score or the price.

3. The fee and seller revenue. Paying Φ reduces initial wealth by Φ and multiplies the buyer's expected utility by $e^{A\Phi}$. The marginal investor is therefore indifferent when $e^{A\Phi}\sqrt{V_1/V_2} = 1$, or $\Phi = (2A)^{-1}\ln(V_2/V_1)$, as in (9). This is the indifference fee for interior participation and the highest fee supporting $\lambda = 1$. Since the mass of buyers is $\theta\lambda$, the seller's revenue is $\theta\lambda\Phi(\lambda)$. □

IA.1.3 Proof of Lemma 2

Fix all primitives other than $\rho \in (0,1]$. Use F, D, κ from the proof of Proposition 3, now writing ρ explicitly in $F(M,\rho;\lambda)$ and $D(M,\rho;\lambda)$. Set $\lambda^\dagger = \min\{\overline{\lambda},1\}$, which is independent of ρ by (12).

1. The regime criterion. By Theorem 1 and Step 1 of the proof of Proposition 3, F is strictly increasing in M and has the unique positive root $M(\lambda)$. Thus

$$\text{the equilibrium is constrained} \iff M(\lambda^\dagger) \geqslant \rho/\sigma_s, \tag{IA.5}$$

$$F(M(\lambda),\rho;\lambda) = 0, \tag{IA.6}$$

$$M(\lambda^\dagger) \geqslant \rho/\sigma_s \iff F(\rho/\sigma_s,\rho;\lambda^\dagger) \leqslant 0. \tag{IA.7}$$

It suffices to determine the sign of F at the unconstrained ratio ρ/σ_s.

2. The cutoff in ρ. Let $\Xi = 1/\sigma_y + (\kappa/\sigma_s)(A\sigma_s\sigma_y + \theta\lambda^\dagger)/(A\sigma_s\sigma_y + \theta) > 0$. Equation (A.28) gives $D(\rho/\sigma_s,\rho;\lambda^\dagger) = \rho\Xi$. Both $\alpha = 1 + \theta\lambda^\dagger/(A\sigma_s\sigma_y) > 0$ and $\gamma = \sigma_q^2/(c^2\Xi^2) \geqslant 0$ are independent of ρ, so

$$F(\rho/\sigma_s,\rho;\lambda^\dagger) = G(\rho) \equiv \alpha\rho^2 - (1+\underline{\sigma}^2) - \gamma/\rho^2. \tag{IA.8}$$

Since $G'(\rho) = 2\alpha\rho + 2\gamma/\rho^3 > 0$, G is strictly increasing on $(0,\infty)$. It is negative near zero, including when $\gamma = 0$, and tends to infinity. Its unique positive zero $\bar{\rho}$ satisfies $\bar{\rho}^2 = [1 + \underline{\sigma}^2 + \sqrt{(1+\underline{\sigma}^2)^2 + 4\alpha\gamma}]/(2\alpha)$. Define $\rho^\dagger = \min\{1,\bar{\rho}\} > 0$. Then $G(\rho) \leqslant 0$ if and only if $\rho \leqslant \rho^\dagger$ on the admissible domain $(0,1]$, including the case in which every admissible ρ is constrained. Step 1 therefore gives the claimed characterization. □

IA.1.4 Proof of Lemma 3

Fix all primitives other than $D_1 > 0$. Use F, D from the proof of Proposition 3, now writing D_1 explicitly in $F(M,D_1;\lambda)$ and $D(M,D_1;\lambda)$. Set $\lambda^\dagger = \min\{\overline{\lambda},1\}$, which is independent of D_1 by (12).

1. The regime criterion. By Theorem 1 and the strict increase of F in M established in Step 1 of

the proof of Proposition 3,

$$\text{the equilibrium is constrained} \iff M(\lambda^\dagger; D_1) \geqslant \rho/\sigma_s, \tag{IA.9}$$

$$M(\lambda^\dagger; D_1) \geqslant \rho/\sigma_s \iff F(\rho/\sigma_s, D_1; \lambda^\dagger) \leqslant 0. \tag{IA.10}$$

If $\theta = 1$, D_1 drops out of (11), so the regime is independent of D_1. Set $D_1^\dagger = \infty$ if the equilibrium is constrained and $D_1^\dagger = 0$ otherwise. It remains to consider $\theta < 1$.

2. The cutoff in D_1. Let $\eta = (A\sigma_s\sigma_y + \theta\lambda^\dagger)/(A\sigma_s\sigma_y + \theta)$, $B_0 = 1/\sigma_y + \theta\eta/(A\sigma_s\sigma_y^2) > 0$, and $B_1 = (1-\theta)\eta/\sigma_s > 0$. Equation (A.28) gives $D(\rho/\sigma_s, D_1; \lambda^\dagger) = \rho(B_0 + B_1 D_1)$. With $\Delta = \rho^2[1 + \theta\lambda^\dagger/(A\sigma_s\sigma_y)] - 1 - \underline{\sigma}^2$, evaluating F at the unconstrained ratio yields

$$F(\rho/\sigma_s, D_1; \lambda^\dagger) = H(D_1) \equiv \Delta - \frac{\sigma_q^2}{c^2\rho^2(B_0 + B_1 D_1)^2}. \tag{IA.11}$$

Thus the equilibrium is constrained exactly when $H(D_1) \leqslant 0$. Because $B_1 > 0$ and $\sigma_q > 0$, H is strictly increasing, so this set is an initial interval. If $\Delta \leqslant 0$, then $H(D_1) < 0$ for every $D_1 > 0$, so $D_1^\dagger = \infty$. If $\Delta > 0$, then $H(D_1) \to \Delta > 0$ as $D_1 \to \infty$, and solving $H(D_1) = 0$ gives $D_1^\dagger = \max\{0, [\sigma_q/(c\rho\sqrt{\Delta}) - B_0]/B_1\}$. A zero cutoff means that every admissible $D_1 > 0$ is unconstrained; equality at a positive finite cutoff is constrained. Together with the case $\theta = 1$, this proves the lemma. □

IA.1.5 Proof of Lemma 4

Fix all primitives other than $A > 0$. Use F, D from the proof of Proposition 3, now writing A explicitly, and set $M^* = \rho/\sigma_s$ and $\lambda^\dagger(A) = \min\{\overline{\lambda}(A), 1\}$. The participation target equals $\overline{\lambda}(A)$ for $A \leqslant A_0$ and one for $A \geqslant A_0$.

1. The regime criterion. By Theorem 1 and the strict increase of F in M established in Step 1 of the proof of Proposition 3, the equilibrium is constrained exactly when $G(A) \leqslant 0$, where

$$G(A) \equiv F(M^*; A, \lambda^\dagger(A)) = \rho^2\left(1 + \frac{\theta\lambda^\dagger(A)}{A\sigma_s\sigma_y}\right) - 1 - \underline{\sigma}^2 - \frac{\sigma_q^2}{c^2\rho^2\Xi(A)^2}.$$

Here $\Xi(A) = D(M^*; A, \lambda^\dagger(A))/\rho > 0$ is the boundary coefficient used in the proof of Lemma 2. Substituting into (A.28) gives

$$\Xi(A) = \begin{cases} \dfrac{1}{\sigma_y} + \dfrac{(1+\lambda_0)[\theta/\sigma_y + A(1-\theta)D_1\sigma_y]}{A\sigma_s\sigma_y + \theta}, & 0 < A \leqslant A_0, \\ \dfrac{1}{\sigma_y} + \dfrac{\theta}{A\sigma_s\sigma_y^2} + \dfrac{(1-\theta)D_1}{\sigma_s}, & A > A_0. \end{cases}$$

Both branches agree at A_0, where $\overline{\lambda}(A_0) = 1$, so G is continuous. Every positive zero of G belongs to the constrained region.

2. Monotonicity on each branch. For $A < A_0$, the first term in G is the constant $\rho^2(1+\lambda_0)$. Differentiation gives $\Xi'(A) = \theta(1+\lambda_0)[(1-\theta)D_1\sigma_y - \sigma_s]/(A\sigma_s\sigma_y + \theta)^2$. Since $\sigma_q > 0$, G' has the same sign as $(1-\theta)D_1\sigma_y - \sigma_s$ on this branch. For $A > A_0$, the first term in G strictly decreases, and $\Xi'(A) = -\theta/(A^2\sigma_s\sigma_y^2) < 0$, so G strictly decreases. Moreover, $\lim_{A\to\infty} G(A) \leqslant \rho^2 - 1 - \underline{\sigma}^2 < 0$ because $\rho \leqslant 1$ and $\underline{\sigma} > 0$. Thus sufficiently large A always yields a constrained equilibrium.

3. The cutoffs. Part (i): $(1-\theta)D_1\sigma_y > \sigma_s$. The function G strictly increases up to A_0 and strictly decreases thereafter. If $G(A_0) \leqslant 0$, every $A > 0$ is constrained; set $\underline{A}^\dagger = \bar{A}^\dagger = A_0$. If $G(A_0) > 0$, there is a unique zero $\bar{A}^\dagger > A_0$. On the lower branch, set $\underline{A}^\dagger = 0$ if $G(0+) \geqslant 0$; otherwise, the unique zero lies in $(0, A_0)$ and defines $\underline{A}^\dagger$. Hence the constrained set is $(0, \underline{A}^\dagger] \cup [\bar{A}^\dagger, \infty)$, with unconstrained set $(\underline{A}^\dagger, \bar{A}^\dagger)$.

Part (ii): $(1-\theta)D_1\sigma_y \leqslant \sigma_s$. The function G is continuous and nonincreasing on $(0, \infty)$, and eventually negative. Therefore $\{A > 0 : G(A) > 0\} = (0, A^\dagger)$ for some finite $A^\dagger \geqslant 0$, with $A^\dagger = 0$ when this set is empty. Its complement is exactly the constrained region $A \geqslant A^\dagger$ within $A > 0$. This proves both parts, including the boundary cases. □

IA.1.6 Proof of Lemma 5

Fix all primitives other than $\theta \in (0, 1]$. Use F, D from the proof of Proposition 3, now writing θ explicitly. Set $M^* = \rho/\sigma_s$, $\theta_c = \lambda_0 A\sigma_s\sigma_y$, and $\lambda^\dagger(\theta) = \min\{\theta_c/\theta, 1\}$.

1. The regime criterion. By Theorem 1 and the strict increase of F in M established in Step 1 of

the proof of Proposition 3, the equilibrium is constrained exactly when $G(\theta) \leqslant 0$, where

$$G(\theta) \equiv F(M^*; \theta, \lambda^\dagger(\theta)) = \rho^2 \left(1 + \frac{\theta \lambda^\dagger(\theta)}{A \sigma_s \sigma_y}\right) - 1 - \underline{\sigma}^2 - \frac{\sigma_q^2}{c^2 \rho^2 \Xi(\theta)^2}.$$

As in the preceding proof, $\Xi(\theta) = D(M^*; \theta, \lambda^\dagger(\theta))/\rho > 0$ denotes the boundary coefficient. Equation (A.28) gives

$$\Xi(\theta) = \begin{cases} \dfrac{1}{\sigma_y} + \dfrac{\theta}{A \sigma_s \sigma_y^2} + \dfrac{(1-\theta) D_1}{\sigma_s}, & 0 < \theta \leqslant \min\{\theta_c, 1\}, \\ \dfrac{1}{\sigma_y} + \dfrac{(1+\lambda_0)[\theta/\sigma_y + A \sigma_y (1-\theta) D_1]}{A \sigma_s \sigma_y + \theta}, & \theta_c < \theta \leqslant 1. \end{cases}$$

The second branch is absent if $\theta_c \geqslant 1$; otherwise the branches agree at θ_c, where $\lambda^\dagger = 1$. Thus G is continuous on $(0, 1]$, and $G(0+) \leqslant \rho^2 - 1 - \underline{\sigma}^2 < 0$.

2. The shape of each branch. On the first branch, Ξ is affine with $\Xi' = [1/(A\sigma_y^2) - D_1]/\sigma_s$. Hence $G' = \rho^2/(A\sigma_s\sigma_y) + 2\sigma_q^2 \Xi'/(c^2\rho^2\Xi^3)$ and $G'' = -6\sigma_q^2(\Xi')^2/(c^2\rho^2\Xi^4) \leqslant 0$, so G is concave there. On the second branch, $\theta\lambda^\dagger = \theta_c$, so the first term in G is constant. Differentiating Ξ gives $\Xi' = A(1+\lambda_0)[\sigma_s - D_1\sigma_y(1 + A\sigma_s\sigma_y)]/(A\sigma_s\sigma_y + \theta)^2$. Thus G' has the sign of $\hat{D}_1 - D_1$ on this branch.

3. The cutoffs. Part (i): $D_1 \leqslant \hat{D}_1$. Since $\hat{D}_1 < 1/(A\sigma_y^2)$, the first branch has $\Xi' > 0$ and $G' > 0$, while the second branch, when present, is nondecreasing. Continuity and $G(0+) < 0$ therefore imply that $\{\theta \in (0, 1] : G(\theta) \leqslant 0\} = (0, \theta^\dagger]$ for some $\theta^\dagger \in (0, 1]$. This includes $\theta^\dagger = 1$ when every admissible θ is constrained, and proves part (i).

Part (ii): $D_1 > \hat{D}_1$. Concavity on the first branch makes its positive set an interval. If the second branch is present, it is decreasing and joins continuously, so it cannot create a separate positive interval. Thus the unconstrained set $\{\theta \in (0, 1] : G(\theta) > 0\}$ is a single interval or is empty. If empty, set $\underline{\theta}^\dagger = \bar{\theta}^\dagger = 1$. Otherwise, let $\underline{\theta}^\dagger$ be its left endpoint; it is positive because $G(0+) < 0$. If $G(1) \leqslant 0$, take $\bar{\theta}^\dagger$ to be its right endpoint; if $G(1) > 0$, choose $\bar{\theta}^\dagger > 1$, making the upper constrained interval empty. By continuity, each chosen cutoff in $(0, 1]$ is constrained. These choices give the two constrained intervals stated in part (ii), completing the proof. □

IA.1.7 Proof of Proposition 4

Fix an open interval of D_1 on which the equilibrium remains constrained. We prove strict monotonicity throughout the interval and strict derivative inequalities wherever the equilibrium is differentiable; at a participation transition the corresponding one-sided inequalities hold.

1. Normalization and informativeness. Use $z = -a_s/a_r$ and $k = \theta\lambda/(A\sigma_s\sigma_y)$ from the baseline proofs, and define

$$\bar{k} \equiv \frac{\theta}{A\sigma_s\sigma_y}, \qquad \zeta \equiv \frac{\sigma_s z}{\rho}, \qquad h \equiv \bar{k} + \frac{(1-\theta)D_1\sigma_y}{\sigma_s},$$
$$b \equiv \frac{1+\underline{\sigma}^2-\rho^2}{\rho^2} > 0, \qquad \nu \equiv \frac{\sigma_q^2\sigma_y^2}{c^2\rho^4} > 0. \tag{IA.12}$$

Only h varies with D_1, with $h \geqslant \bar{k}$ and $dh/dD_1 = (1-\theta)\sigma_y/\sigma_s > 0$. Equations (A.28), (A.30), and (A.34) give the normalized inverse loading, the binding noise constraint, and the seller's revenue up to a positive constant:

$$\mathcal{D}(k,\zeta;h) \equiv \frac{\sigma_y}{\rho a_r} = 1 + h\frac{\zeta(\zeta+k)}{\zeta+\bar{k}}, \tag{IA.13}$$

$$\mathcal{G}(k,\zeta;h) \equiv (k\zeta - b)\mathcal{D}(k,\zeta;h)^2 = \nu, \tag{IA.14}$$

$$\widehat{\Pi}(k,\zeta) \equiv k\ln\left(1 + \frac{1}{k(k+\zeta+\zeta^{-1})}\right). \tag{IA.15}$$

Here $0 < k \leqslant \bar{k}$. The conditional noise choice in the proof of Theorem 1 implies $\zeta \geqslant 1$. Equality at an equilibrium attains the relaxed optimum at its noise floor and therefore marks the boundary of the constrained regime; inside the stated interval, $\zeta > 1$. Write $x \equiv k\zeta$ and $\omega \equiv x - b > 0$. By (A.25)–(A.26),

$$\mathrm{Var}(g_r) = \rho^2(1+x), \qquad \mathrm{corr}(g_r,\tilde{g}) = \frac{1}{\rho\sqrt{1+x}}, \tag{IA.16}$$

$$\mathrm{corr}(p,\tilde{g}) = \frac{1}{\rho\sqrt{1+x+\zeta^2}}. \tag{IA.17}$$

It suffices to show that x and ζ strictly decrease with h.

2. The seller's first-order condition. For $\zeta > 1$, define the slope ratios

$$\mathcal{H} \equiv \frac{\widehat{\Pi}_k}{\widehat{\Pi}_\zeta}, \qquad \mathcal{T} \equiv \frac{\mathcal{G}_k}{\mathcal{G}_\zeta} > 0. \tag{IA.18}$$

We have $\widehat{\Pi}_\zeta < 0$ and $\mathcal{G}_k, \mathcal{G}_\zeta, \mathcal{G}_h > 0$. Subscripts hold the other arguments fixed; a subscript $\zeta \mid x$ instead holds $x = k\zeta$ fixed. With $n \equiv k(k + \zeta + \zeta^{-1})$, differentiation gives

$$\begin{aligned} \mathcal{H}_k &= \frac{2k\{1 - n\ln(1+n^{-1})\} + (\zeta + \zeta^{-1})\ln(1+n^{-1})}{k^2(1-\zeta^{-2})} > 0, \\ \mathcal{H}_\zeta &= \frac{2 - (2n+1)\ln(1+n^{-1})}{k} - \frac{2\mathcal{H}}{\zeta(\zeta^2-1)} < 0 \quad \text{if } \mathcal{H} > 0. \end{aligned} \tag{IA.19}$$

The signs use $n\ln(1+n^{-1}) < 1$ and $\ln(1+n^{-1}) > 2/(2n+1)$. For the constraint slope, rearranging (IA.14) yields

$$\begin{aligned} \frac{1}{\mathcal{T}} &= \chi\frac{x}{\zeta^2} + (1-\chi)\left(1 + \frac{\bar{k}}{\zeta+\bar{k}} + \frac{\bar{k}x}{\zeta^2(\zeta+\bar{k})}\right), \\ \chi &\equiv \frac{\zeta + \bar{k} + h(\zeta^2 + x)}{\zeta + \bar{k} + h(\zeta^2 + x + 2\omega)} \in (0,1). \end{aligned} \tag{IA.20}$$

The second term being averaged exceeds the first by $(\zeta^2 + 2\bar{k}\zeta - x)/[\zeta(\zeta+\bar{k})] > 0$, since $x \leqslant \bar{k}\zeta$. Both terms increase with x and decrease with ζ at fixed x, while $\chi_x < 0$, $\chi_{\zeta|x} > 0$, and $\chi_h < 0$. Thus

$$\mathcal{T}_k < 0, \qquad \mathcal{T}_{\zeta|x} > 0, \qquad \mathcal{T}_h < 0. \tag{IA.21}$$

Solve the binding constraint for $k = k(\zeta, h)$, holding ν fixed. Then $k_\zeta = -1/\mathcal{T}$ and

$$\mathcal{G}_{\zeta|x} = 2\omega\mathcal{D}\,h\frac{\zeta^2 + 2\bar{k}\zeta - x}{(\zeta+\bar{k})^2} > 0, \qquad x_\zeta\big|_{\mathcal{G}=\nu} = -\frac{\zeta\mathcal{G}_{\zeta|x}}{\mathcal{G}_k} < 0.$$

At an interior optimum, differentiating revenue along this constraint gives $\widehat{\Pi}_\zeta(1 - \mathcal{H}/\mathcal{T}) = 0$. Define $\mathcal{F}(\zeta, h) \equiv \mathcal{H}(k(\zeta,h),\zeta) - \mathcal{T}(k(\zeta,h),\zeta;h)$, so the first-order condition is $\mathcal{F} = 0$. At any such point $\mathcal{H} = \mathcal{T} > 0$, and the signs above imply

$$\mathcal{F}_\zeta = \mathcal{H}_\zeta - \frac{k}{\zeta}\mathcal{H}_k - \mathcal{T}_{\zeta|x} + \frac{\mathcal{H}_k - \mathcal{T}_k}{\zeta}x_\zeta\big|_{\mathcal{G}=\nu} < 0. \tag{IA.22}$$

Revenue along the constraint has derivative $-\widehat{\Pi}_\zeta \mathcal{F}/\mathcal{T}$, so every stationary point is a strict maximum. Consequently there is at most one interior maximum, and it cannot coexist with a maximum at the participation cap. If the constraint meets $\zeta = 1$ before the cap, its participation there exceeds the relaxed optimum λ_0 in normalized units; hence revenue increases when moving from that endpoint onto $\zeta > 1$. The baseline existence argument therefore gives a unique constrained optimum. It is continuous in h: the feasible frontier is continuous, and $\widehat{\Pi}(k, \zeta) < 1/\zeta$ bounds its tail uniformly.

3. Endogenous participation. Suppose $k < \bar{k}$. On the binding constraint, let

$$\mathcal{L} \equiv \frac{\mathcal{G}_h}{\mathcal{G}_k} = \frac{2\omega(\zeta^2 + x)}{\zeta[\zeta + \bar{k} + h(\zeta^2 + x + 2\omega)]} > 0, \qquad k_h = -\mathcal{L}.$$

Differentiating this expression gives $\mathcal{L}_x > 0$: its numerator after removing a positive factor is $(\zeta + \bar{k})(\zeta^2 + x + \omega) + h[(\zeta^2 + x)^2 + 2\omega^2] > 0$. The derivative $\mathcal{L}_{\zeta|x}$ has the sign of

$$\bar{k}(\zeta^2 - x) - 2\zeta x - h[\zeta^4 + 2b(\zeta^2 - x) + 3x^2] < 0.$$

If $x \leqslant \zeta^2$, this follows from $h \geqslant \bar{k}$, $\zeta \geqslant 1$, and the bracket being at least ζ^4. If $x > \zeta^2$, the first two terms are negative and $b < x$ makes the bracket greater than $(\zeta^2 + x)^2 > 0$. Since $x_\zeta|_{\mathcal{G}=\nu} < 0$, the total derivative of $\mathcal{L}$ along the constraint is negative. Equality of the mixed derivatives of $k(\zeta, h)$, using $k_h = -\mathcal{L}$ and $k_\zeta = -1/\mathcal{T}$, consequently gives

$$0 < k_{h\zeta} = \frac{\mathcal{T}_h - \mathcal{T}_k \mathcal{L}}{\mathcal{T}^2}, \qquad \mathcal{F}_h = -\mathcal{H}_k \mathcal{L} - (\mathcal{T}_h - \mathcal{T}_k \mathcal{L}) < 0.$$

By (IA.22) and the implicit function theorem,

$$\frac{d\zeta^*}{dh} = -\frac{\mathcal{F}_h}{\mathcal{F}_\zeta} < 0.$$

Differentiating $\mathcal{H} = \mathcal{T}$ along the optimum and using $x = k\zeta$ also yields

$$\frac{dx^*}{dh} = \frac{\zeta \mathcal{T}_h + \left(k\mathcal{H}_k - \zeta \mathcal{H}_\zeta + \zeta \mathcal{T}_{\zeta|x}\right) d\zeta^*/dh}{\mathcal{H}_k - \mathcal{T}_k} < 0.$$

Every sign follows from Step 2. In particular, no sign restriction on $d\lambda^*/dD_1$ is required.

4. Full participation and transitions. At $k = \bar{k}$, the binding constraint reduces to

$$(\bar{k}\zeta - b)(1 + h\zeta)^2 = \nu. \tag{IA.23}$$

Its derivative gives

$$\frac{d\zeta^*}{dh} = -\frac{\mathcal{G}_h}{\mathcal{G}_\zeta} < 0, \qquad \frac{dx^*}{dh} = \bar{k}\frac{d\zeta^*}{dh} < 0. \tag{IA.24}$$

At a transition between the cap and an interior optimum, (IA.22) gives a smooth continuation of the interior stationary solution. The cap solution is smooth as well. Locally, the equilibrium ratio is the larger of the cap and interior ratios, while x is the smaller of their corresponding products because x decreases along the binding constraint. Both branches have strictly negative derivatives of x and ζ, including their limiting derivatives at the transition. Hence x and ζ are strictly decreasing throughout the constrained interval, with negative one-sided derivatives at any participation transition. Equations (IA.16)–(IA.17) and $dh/dD_1 > 0$ then establish the claimed strict increases in both correlations, with strictly positive derivatives wherever differentiable. □

IA.1.8 Proof of Part 2 of Proposition 3

1. The normalized problem. Use the normalization and slope ratios in Steps 1–2 of Section IA.1.7, namely (IA.12)–(IA.15) and (IA.18). In the present comparative static, $\bar{k}, h, b$ are fixed and only $\nu = \sigma_q^2\sigma_y^2/(c^2\rho^4)$ varies. Those steps' identities and slope signs require only $h \geqslant \bar{k}$ and therefore include $\theta = 1$, when $h = \bar{k}$. Work first at $\sigma_q > \sigma_q^\dagger$, so the optimum satisfies $\zeta > 1$, $k\zeta > b$, and $\mathcal{G}(k, \zeta; h) = \nu$. By (IA.16)–(IA.17), it suffices to show that ζ^* and $x^* = k^*\zeta^*$ strictly increase with ν; these are positive constant multiples of z and $\lambda^* z$, respectively.

2. Endogenous participation. Suppose $0 < k^* < \bar{k}$. Holding h fixed, solve the binding constraint for $k = k(\zeta, \nu)$, giving $k_\nu = 1/\mathcal{G}_k > 0$. Write $\mathcal{F}(\zeta, h; \nu)$ for the reduced first-order condition in Step 2 of Section IA.1.7, making its dependence on ν explicit. The functions $\mathcal{H}$ and $\mathcal{T}$ have no direct

dependence on ν, so

$$\mathcal{F}_\nu = \frac{\mathcal{H}_k - \mathcal{T}_k}{\mathcal{G}_k} > 0, \qquad \mathcal{F}_\zeta < 0,$$

where the signs follow from (IA.19), (IA.21), and (IA.22). The implicit function theorem consequently gives a smooth local optimum and

$$\frac{d\zeta^*}{d\nu} = -\frac{\mathcal{F}_\nu}{\mathcal{F}_\zeta} > 0.$$

Differentiating the stationary equation $\mathcal{H} = \mathcal{T}$, using $x = k\zeta$ and holding h fixed, gives

$$\frac{dx^*}{d\nu} = \frac{k\mathcal{H}_k - \zeta\mathcal{H}_\zeta + \zeta\mathcal{T}_{\zeta|x}}{\mathcal{H}_k - \mathcal{T}_k}\frac{d\zeta^*}{d\nu} > 0.$$

Indeed, at the optimum $\mathcal{H} = \mathcal{T} > 0$, and Step 2 of Section IA.1.7 establishes $\mathcal{H}_k > 0$, $\mathcal{H}_\zeta < 0$, $\mathcal{T}_k < 0$, and $\mathcal{T}_{\zeta|x} > 0$.

3. Full participation and boundaries. At $k^* = \bar{k}$, differentiating $\mathcal{G}(\bar{k}, \zeta^*; h) = \nu$ yields

$$\frac{d\zeta^*}{d\nu} = \frac{1}{\mathcal{G}_\zeta} > 0, \qquad \frac{dx^*}{d\nu} = \bar{k}\frac{d\zeta^*}{d\nu} > 0.$$

The uniqueness argument in Step 2 of Section IA.1.7 applies for each ν, and the uniform revenue bound $\widehat{\Pi} < 1/\zeta$ also gives continuity as ν varies. At a participation transition, the cap solution and the interior stationary solution have smooth local continuations. As in Step 4 of that proof, the equilibrium ratio is locally the larger of the two ratios, while x is the smaller of their corresponding products. Both branches have strictly positive derivatives of ζ and x, so these two objects remain strictly increasing across the transition, even if their ordinary derivatives do not exist there. If $\sigma_q^\dagger > 0$, continuity to the unique relaxed optimum at $\sigma_q = \sigma_q^\dagger$ extends these strict increases to the constrained-regime boundary. If $\sigma_q^\dagger = 0$, every admissible $\sigma_q > 0$ is already covered. Since $d\nu/d\sigma_q = 2\sigma_q\sigma_y^2/(c^2\rho^4) > 0$, both z and $\lambda^* z$ strictly increase with σ_q. Equations (IA.16)–(IA.17) establish the claimed strict decreases in score and price informativeness. □

IA.1.9 Proof of Remark 1

Use $z = -a_s/a_r$ and $\Pi(\lambda, z)$ from the proof of Theorem 1.

1. Unconstrained equilibria. Independence of the shocks gives

$$\operatorname{corr}(g_r, \tilde{g}) = \operatorname{Var}(g_r)^{-1/2}, \qquad \operatorname{corr}(p, \tilde{g}) = \left[\operatorname{Var}(g_r) + \sigma_s^2 z^2\right]^{-1/2}. \tag{IA.25}$$

In the unconstrained equilibrium, Theorem 1 implies $\lambda^* = \min\{\overline{\lambda}, 1\}$ and

$$z = M^* = \frac{\rho}{\sigma_s}, \qquad \operatorname{Var}(g_r) = \rho^2 \left(1 + \frac{\theta\lambda^*}{A\sigma_s\sigma_y}\right), \tag{IA.26}$$

$$\operatorname{corr}(p, \tilde{g}) = \left[\operatorname{Var}(g_r) + \rho^2\right]^{-1/2}. \tag{IA.27}$$

Neither λ^* nor these expressions depends on D_1, proving the unconstrained claim.

2. No manipulation. When $q \equiv 0$, the revenue formula (A.33) and feasibility equation (A.30) reduce the seller's problem to

$$\max_{0<\lambda\leqslant 1,\ \sigma_u \geqslant \underline{\sigma}} \Pi(\lambda, z), \tag{IA.28}$$

$$\rho^2 + \frac{\rho\theta\lambda}{A\sigma_y} z = 1 + \sigma_u^2. \tag{IA.29}$$

Both the objective and the feasible set are independent of D_1. Hence so is the equilibrium correspondence for (λ, z, σ_u). Since $\operatorname{Var}(g_r) = 1 + \sigma_u^2$, (IA.25) gives the same conclusion for score and price informativeness. □

IA.1.10 Proof of Proposition 5

Fix $\sigma_q \geqslant \sigma_q^{\text{full}}$. Step 3 of the proof of Proposition 3 establishes $\lambda^* = 1$ and $\sigma_u^* = \underline{\sigma}$ uniformly over $\theta \in (0, 1]$. Write $M = M(1)$ along this equilibrium as θ varies.

1. Parameterizing the equilibrium. Equations (11) and (A.28) give

$$\frac{\rho\theta}{A\sigma_y} M = 1 + \underline{\sigma}^2 - \rho^2 + a_r^2 \frac{\sigma_q^2}{c^2}, \qquad \frac{1}{a_r} = \frac{\rho}{\sigma_y} + \left(\frac{\theta}{A\sigma_y^2} + (1-\theta)D_1\right) M. \tag{IA.30}$$

Eliminating θM yields

$$M = \frac{1}{D_1}\left[\frac{1}{a_r} - \frac{\rho}{\sigma_y} + \frac{A\sigma_y}{\rho}\left(D_1 - \frac{1}{A\sigma_y^2}\right)\left(1 + \underline{\sigma}^2 - \rho^2 + a_r^2\frac{\sigma_q^2}{c^2}\right)\right]. \tag{IA.31}$$

The binding equation has $F_M > 0$, so its solution is smooth for $\theta > 0$. Differentiating the implied $\theta(a_r)$ gives

$$\frac{d\theta}{da_r} = \frac{A\sigma_y}{\rho D_1 M^2}\left[\frac{1 + \underline{\sigma}^2 - \rho^2}{a_r^2} + \frac{\sigma_q^2}{c^2}\left(3 - \frac{2\rho a_r}{\sigma_y}\right)\right] > 0,$$

since $0 < a_r < \sigma_y/\rho$. The first identity in (IA.30) implies $M \to \infty$ as $\theta \downarrow 0$, and the second then implies $a_r \to 0$. Thus a_r increases continuously from zero to its value at $\theta = 1$.

2. At most one interior maximum. By independence of the shocks, price informativeness equals $J(a_r)^{-1/2}$, where $J(a_r) = 1 + \underline{\sigma}^2 + a_r^2\sigma_q^2/c^2 + \sigma_s^2 M(a_r)^2$. Primes in this step denote derivatives with respect to a_r. Equation (IA.31) gives

$$M'' = \frac{2}{D_1 a_r^3}\left[1 + \frac{A\sigma_y}{\rho}\left(D_1 - \frac{1}{A\sigma_y^2}\right)\frac{\sigma_q^2}{c^2}a_r^3\right] > 0.$$

Positivity is immediate if $D_1 \geqslant 1/(A\sigma_y^2)$; otherwise, $M > 0$ in (IA.31) makes the bracket exceed $\rho a_r/\sigma_y > 0$. Consequently, $J'' = 2\sigma_q^2/c^2 + 2\sigma_s^2[(M')^2 + MM''] > 0$. Also, $M \sim 1/(D_1 a_r)$ as $a_r \downarrow 0$, so $J' \to -\infty$. Strict convexity therefore gives an interior minimum of J exactly when its derivative at $\theta = 1$ is positive; otherwise $J' < 0$ throughout $\theta \in (0, 1)$. Because a_r increases with θ, these are precisely the two possible patterns for price informativeness.

3. The cutoff. Let M_1 denote M at $\theta = 1$. It is the unique root above $A\sigma_y(1 + \underline{\sigma}^2 - \rho^2)/\rho$ of

$$\left[\frac{\rho M_1}{A\sigma_y} - (1 + \underline{\sigma}^2 - \rho^2)\right]\left(\frac{\rho}{\sigma_y} + \frac{M_1}{A\sigma_y^2}\right)^2 = \frac{\sigma_q^2}{c^2}. \tag{IA.32}$$

The left-hand side is zero at that bound and strictly increases to infinity above it; in particular, M_1 is independent of D_1. Define

$$D_{\text{crit}}(\sigma_q) = \frac{\sigma_s^2 M_1}{\frac{\rho}{A\sigma_y} + 2\sigma_s^2 M_1}\left[\frac{\rho c^2}{A\sigma_y\sigma_q^2}\left(\frac{\rho}{\sigma_y} + \frac{M_1}{A\sigma_y^2}\right)^3 + \frac{2}{A\sigma_y^2}\right] > 0. \tag{IA.33}$$

Substituting $a_r = (\rho/\sigma_y + M_1/(A\sigma_y^2))^{-1}$ into the derivative of J gives $\operatorname{sgn} J'(a_r)|_{\theta=1} = \operatorname{sgn}[D_1 - D_{\text{crit}}(\sigma_q)]$. If $D_1 \leqslant D_{\text{crit}}$, strict convexity yields $J' < 0$ at every $\theta \in (0,1)$, including when equality holds at the endpoint. If $D_1 > D_{\text{crit}}$, J' crosses zero exactly once. Finally, $d\operatorname{corr}(p,\tilde{g})/d\theta = -J'(a_r)(da_r/d\theta)/(2J^{3/2})$, which gives the stated derivative signs and the unique interior maximum. □

IA.1.11 Investor composition in constrained equilibrium

The following proposition extends the comparative statics in Proposition 5 without imposing full participation. Its lower cutoff is $\hat{D}_1$ from Lemma 5; the two cutoffs are sufficient conditions, rather than a complete characterization for intermediate values of D_1.

Proposition IA.1 (Investor composition and price informativeness)**.** *Suppose $\sigma_q > 0$ and fix all primitives other than θ.*

(i) If $D_1 < \hat{D}_1 \equiv \sigma_s/[\sigma_y(1 + A\sigma_s\sigma_y)]$, price informativeness $\operatorname{corr}(p,\tilde{g})$ is strictly increasing in θ on any open interval over which the equilibrium remains constrained.

(ii) If $D_1 \geqslant 1/(A\sigma_y^2)$, price informativeness is strictly decreasing in θ on any open interval over which the equilibrium remains constrained with $0 < \lambda^* < 1$.

The corresponding derivative inequalities are strict wherever the equilibrium is differentiable.

Proof. 1. The normalized problem. Use $(\bar{k}, k, \zeta, h, b, \nu)$, $x = k\zeta$, $\omega = x - b > 0$, and $(\mathcal{D}, \mathcal{G}, \hat{\Pi}, \mathcal{H}, \mathcal{T})$ from (IA.12)–(IA.18). Define

$$e \equiv 1 - AD_1\sigma_y^2, \qquad d \equiv \frac{D_1\sigma_y}{\sigma_s}, \qquad h = d + e\bar{k}, \qquad \frac{d\bar{k}}{d\theta} = \frac{1}{A\sigma_s\sigma_y} > 0. \tag{IA.34}$$

Thus part (i) imposes $e > d \geqslant 0$, whereas part (ii) imposes $e \leqslant 0$ and $d > 0$. Write $Y \equiv x + \zeta^2$, so (IA.17) gives $\operatorname{corr}(p,\tilde{g}) = 1/[\rho\sqrt{1+Y}]$. It suffices to establish the opposite signs for the change in Y.

First consider a constrained optimum with $k < \bar{k}$ and $\zeta > 1$. At this optimum $\mathcal{H} = \mathcal{T} > 0$, and

(IA.22) gives a strict second-order condition. Moreover,

$$k > \lambda_0 > \frac{13}{20}, \qquad k < \zeta, \qquad 0 < \mathcal{H} < \frac{\zeta}{k}, \qquad 0 < \omega < x < \zeta^2. \tag{IA.35}$$

To verify the first two bounds, put $n = k(k + \zeta + \zeta^{-1})$ as in (IA.19). Direct differentiation gives

$$\widehat{\Pi}_k = \frac{n(n+1)\ln(1+n^{-1}) - n - k^2}{n(n+1)}, \qquad 0 < n(n+1)\ln(1+n^{-1}) - n = n\int_0^1 \frac{1-u}{n+u}\,du < \frac{1}{2}.$$

The integral expression is strictly increasing in n. Since $n > k(k+2)$ and $\widehat{\Pi}_k < 0$, the baseline first-order condition for λ_0 implies $k > \lambda_0$. That condition also gives $\lambda_0 > 13/20$: its left-hand side at $13/20$ is $\ln(1089/689) - 800/1749 > 0$, using $\ln[(1+t)/(1-t)] > 2(t + t^3/3)$ at $t = 200/889$. The displayed bound and $\mathcal{H} = [k^2 - n(n+1)\ln(1+n^{-1}) + n]/[k^2(1-\zeta^{-2})]$ imply $\mathcal{H}(\zeta, \zeta) > 1$. Because $\mathcal{H}_k > 0$ and $\mathcal{T} < \zeta/k$ by (IA.20), $k \geqslant \zeta$ would contradict $\mathcal{H} = \mathcal{T}$. This proves (IA.35).

2. A bound on the revenue slope. In coordinates (Y, ζ), let

$$\mathscr{P}(Y,\zeta) \equiv \widehat{\Pi}\left(\frac{Y-\zeta^2}{\zeta}, \zeta\right), \qquad W \equiv -\frac{\mathscr{P}_Y}{\mathscr{P}_\zeta} = \frac{1}{2\zeta + k - \zeta/\mathcal{H}}.$$

At the optimum, $\mathscr{P}_\zeta > 0$ and $W > 0$. Indeed, (IA.20) gives $k/\zeta < 1/\mathcal{T} < 2 + k/\zeta$, and $\mathcal{H} = \mathcal{T}$. We will use the following strict inequality, where W_ζ holds Y fixed:

$$W_\zeta > \frac{W}{\zeta(\zeta-1)}. \tag{IA.36}$$

Here is an algebraic verification. Set $v = 1 - \zeta^{-2} > 0$. The derivative in (IA.19) has the integral representation

$$\mathcal{H}_k = \frac{1}{k^3 v}\int_0^1 \frac{(1-u)[2n^2 + (n-k^2)u]}{(n+u)^2}\,du > \frac{n^2}{k^3 v(n+1/3)^2} > \frac{7}{10k^3 v}.$$

The first inequality uses Jensen's inequality with density $2(1-u)$; the second uses $n > (13/20)(53/20) = 689/400$. The companion derivative in Step 2 of Section IA.1.7 also gives $-\mathcal{H}_\zeta > 2\mathcal{H}/[\zeta(\zeta^2-1)]$.

Differentiating W at fixed Y, so $k_\zeta = -2 - k/\zeta$, shows that (IA.36) is equivalent to

$$\zeta^2[(\zeta-1)\{(2+k/\zeta)\mathcal{H}_k - \mathcal{H}_\zeta\} + \mathcal{H}] + \mathcal{H}^2[k(\zeta-2) - 2\zeta] > 0.$$

Substitute the two lower bounds just obtained. The resulting expression is a quadratic in $\mathcal{H}$ with positive constant and linear coefficients. If its quadratic coefficient is nonnegative, it is positive. Otherwise, concavity and $0 < \mathcal{H} < \zeta/k$ reduce the verification to the two endpoints. At $\mathcal{H} = 0$ it is positive; at $\mathcal{H} = \zeta/k$ it equals

$$\frac{\zeta^3[20k^2\zeta - 20k\zeta - 13k + 14\zeta]}{10k^3(\zeta+1)} > 0.$$

For $k \leqslant 1$, the bracket exceeds $20k^2 - 33k + 14 > 0$, since $\zeta > 1$. For $k \geqslant 1$, it exceeds $k(20k^2 - 20k + 1) > 0$, since $\zeta > k$. This proves (IA.36).

3. The response of the binding constraint. In the same coordinates, $\omega = Y - \zeta^2 - b$ and

$$\mathcal{G} = \omega\left(1 + \frac{hY}{\zeta + \bar{k}}\right)^2 = \nu.$$

Its derivative with respect to ζ at fixed Y is negative, while its derivative with respect to Y at fixed ζ is positive. Hence it defines a smooth function $\zeta(Y, \bar{k})$ along the path (IA.34). Put $S \equiv \zeta_Y > 0$ and

$$N \equiv \zeta(\zeta+\bar{k})^2 + hY(\zeta^2 + \bar{k}\zeta + \omega) > 0,$$
$$R \equiv \frac{N}{\zeta\omega Y} = \frac{(\zeta+\bar{k})^2}{\omega Y} + \frac{h(\zeta+\bar{k})}{\omega} + \frac{h}{\zeta} > 0.$$

Implicit differentiation at fixed Y gives

$$\zeta_{\bar{k}} = \frac{e\zeta - d}{\zeta R}. \tag{IA.37}$$

We also need $R_Y > 0$, where the derivative follows $\zeta = \zeta(Y, \bar{k})$ with $\bar{k}$ fixed. A direct calculation

gives

$$R_Y = \frac{(\bar{k}+\zeta)(\mathcal{A}_0 + h\mathcal{A}_1 + h^2\mathcal{A}_2)}{2\zeta^2\omega Y^2 N},$$

where

$$\begin{aligned}
\mathcal{A}_0 &= -2\zeta^2(\bar{k}+\zeta)^2(\bar{k}\zeta - x),\\
\mathcal{A}_1 &= (\bar{k}+\zeta)Y[2\bar{k}\zeta^3 + 3\zeta^4 + \zeta^2 x + \omega(\zeta^2 - x)],\\
\mathcal{A}_2 &= Y^2[4\bar{k}\zeta^3 + 2(x^2-\omega^2) + (\zeta^2 - x)(5x+\omega) + 3(\zeta^2 - x)^2].
\end{aligned}$$

By (IA.35), $\mathcal{A}_2 > 0$. Also, $h \geqslant \bar{k}$, $Y > \zeta^2$, $\zeta > 1$, and $0 < x \leqslant \bar{k}\zeta$ imply $h\mathcal{A}_1 + \mathcal{A}_0 > \bar{k}(\bar{k}+\zeta)\zeta^4 > 0$. Thus $R_Y > 0$ at the optimum.

4. Endogenous participation. Define revenue along the binding constraint by $\mathscr{V}(Y,\bar{k}) \equiv \mathscr{P}(Y,\zeta(Y,\bar{k}))$. Its first-order condition is

$$\mathscr{V}_Y = \mathscr{P}_\zeta(S - W) = 0.$$

The strict second-order condition (IA.22) is preserved by this change of coordinate, so $\mathscr{V}_{YY} < 0$. Differentiating the first-order condition at the optimum, and using (IA.37) and equality of mixed derivatives of ζ, gives

$$\mathscr{V}_{Y\bar{k}} = \mathscr{P}_\zeta\left[\frac{dS}{\zeta^2 R} - \zeta_{\bar{k}}\left(\frac{R_Y}{R} + W_\zeta\right)\right], \qquad \frac{dY^*}{d\bar{k}} = -\frac{\mathscr{V}_{Y\bar{k}}}{\mathscr{V}_{YY}}. \tag{IA.38}$$

For part (i), $e > d \geqslant 0$ and $\zeta > 1$ imply $\zeta_{\bar{k}} > 0$. Along the binding constraint,

$$\frac{(\zeta_{\bar{k}})_Y}{\zeta_{\bar{k}}} = \frac{dS}{\zeta(e\zeta - d)} - \frac{R_Y}{R} < \frac{S}{\zeta(\zeta-1)} = \frac{W}{\zeta(\zeta-1)} < W_\zeta,$$

where $S = W$ at the optimum and the last inequality is (IA.36). Consequently $\mathscr{V}_{Y\bar{k}} = \mathscr{P}_\zeta\zeta_{\bar{k}}[(\zeta_{\bar{k}})_Y/\zeta_{\bar{k}} - W_\zeta] < 0$, so $dY^*/d\bar{k} < 0$. For part (ii), $e \leqslant 0$ and $d > 0$ imply $\zeta_{\bar{k}} < 0$. Every term in the brackets in (IA.38) is then positive, because $S, R, R_Y, W_\zeta > 0$. Thus $dY^*/d\bar{k} > 0$. Equation (IA.34) gives the corresponding signs with respect to θ.

5. Full participation and transitions. For part (i), it remains to consider $k = \bar{k}$. Here $\mathcal{D} = 1 + h\zeta$,

$\omega = \bar{k}\zeta - b$, and $Y = \bar{k}\zeta + \zeta^2$. Differentiating (IA.23) along (IA.34) yields

$$\frac{dY}{d\bar{k}} = \frac{2\zeta[-\zeta\mathcal{D} + \omega(d - 2e\zeta)]}{\bar{k}\mathcal{D} + 2h\omega} < 0,$$

because $e > d \geqslant 0$ and $\zeta \geqslant 1$. The uniqueness argument and the bound $\widehat{\Pi} < 1/\zeta$ in Step 2 of Section IA.1.7 apply pointwise and locally uniformly as both $\bar{k}$ and h vary. Hence the constrained optimum is continuous in θ. At a participation transition with $\zeta > 1$, (IA.22) gives a smooth continuation of the interior stationary branch, while the cap branch is smooth by (IA.23). Along the binding constraint, Y increases with ζ because $S > 0$, so the equilibrium Y is locally the larger of these two branches. Both have strictly negative derivatives in part (i), including their limits at the transition. Strict monotonicity therefore survives a participation kink.

Finally, $\zeta = 1$ at an equilibrium implements the relaxed optimum at the noise floor. In part (i), the regime characterization in Lemma 5, with $D_1 < \hat{D}_1$, places such points at the boundary of a constrained interval. In part (ii), a relaxed optimum with partial participation has $k = \lambda_0$ and $\mathcal{G}(\lambda_0, 1) = (\lambda_0 - b)[1 + h(1 + \lambda_0)/(1 + \bar{k})]^2$. Along (IA.34), $d[h/(1 + \bar{k})]/d\bar{k} = (e - d)/(1 + \bar{k})^2 < 0$, and equality to $\nu > 0$ requires $\lambda_0 > b$. The displayed relaxed-floor value therefore decreases strictly at any such equality, which can occur only at a regime boundary. Continuity extends the strict monotonicity to any included boundary points. Since price informativeness is strictly decreasing in Y, these conclusions prove both parts. □

IA.1.12 Proof of Remark 2

1. The unconstrained price ratio. By Theorem 1, the unconstrained equilibrium implements $-a_s/a_r = M^* = \rho/\sigma_s$. Hence (IA.25) gives

$$\operatorname{corr}(p, \tilde{g}) = \left[\operatorname{Var}(g_r) + \rho^2\right]^{-1/2}. \tag{IA.39}$$

2. The two cases. With partial participation, (12) and (13) imply $\operatorname{Var}(g_r) = \rho^2(1 + \lambda_0)$, so price informativeness is $1/(\rho\sqrt{2 + \lambda_0})$, independent of θ. With full participation, (14) gives $\operatorname{Var}(g_r) =$

$\rho^2[1+\theta/(A\sigma_s\sigma_y)]$, so price informativeness is $1/(\rho\sqrt{2+\theta/(A\sigma_s\sigma_y)})$, strictly decreasing in θ. □

IA.1.13 Proof of Lemma 6

Fix the market belief $\tilde{I}$ and use r_1, r_2, V_1, V_2 from the proof of Proposition 1.

1. Manipulation and beliefs. For a price $p = a_0 + a_r g_r + a_s \tilde{s}$, the manager's marginal return to manipulation is qa_r. Strict concavity therefore gives

$$\delta^*(q) = a_r q/c, \tag{IA.40}$$

$$g_r = \tilde{g} + I + u + a_r q/c. \tag{IA.41}$$

For fixed price coefficients, I shifts only the score mean. Under investors' belief, the score has mean $\tilde{I} + a_r\bar{q}/c$; let $\bar{p}$ denote the expected price under that belief. The baseline updating formulas become

$$r_1^{-1} = 1 + \sigma_u^2 + a_r^2\sigma_q^2/c^2, \qquad r_2^{-1} = r_1^{-1} + (a_s/a_r)^2\sigma_s^2, \tag{IA.42}$$

$$\mathbb{E}[\tilde{y} \mid \mathscr{F}_I; \tilde{I}] = \bar{y} + \rho\sigma_y\tilde{I} + \rho\sigma_y r_1(g_r - \tilde{I} - a_r\bar{q}/c), \tag{IA.43}$$

$$\mathbb{E}[\tilde{y} \mid \mathscr{F}_U; \tilde{I}] = \bar{y} + \rho\sigma_y\tilde{I} + \rho\sigma_y r_2(p - \bar{p})/a_r, \tag{IA.44}$$

$$\mathrm{Var}[\tilde{y} \mid \mathscr{F}_I] = V_1, \qquad \mathrm{Var}[\tilde{y} \mid \mathscr{F}_U] = V_2. \tag{IA.45}$$

Thus the belief shifts means and leaves conditional variances unchanged.

2. Price and seller's choice. With $\phi_i = (\mathbb{E}[\tilde{y} \mid \mathscr{F}_i; \tilde{I}] - p)/(AV_i)$, market clearing is

$$\theta\{\lambda\phi_I + (1-\lambda)\phi_U\} + (1-\theta)(D_0 - D_1 p) = \tilde{s}. \tag{IA.46}$$

Holding $\tilde{I}$ fixed and taking differentials eliminates the mean shifts:

$$\left[\theta\left(\frac{\lambda}{V_1} + \frac{(1-\lambda)(1-\rho\sigma_y r_2/a_r)}{V_2}\right) + A(1-\theta)D_1\right]dp = \theta\frac{\lambda\rho\sigma_y r_1}{V_1}dg_r - A\,d\tilde{s}. \tag{IA.47}$$

This is (A.21), so the unique loadings at each (λ, σ_u) satisfy

$$(a_r, a_s) = (a_r^{\text{base}}, a_s^{\text{base}}). \tag{IA.48}$$

Taking expectations in (IA.46) then gives

$$\bar{p}(\tilde{I}) = \frac{\theta(\lambda/V_1 + (1-\lambda)/V_2)(\bar{y} + \rho\sigma_y \tilde{I}) + A(1-\theta)D_0 - A\bar{s}}{\theta(\lambda/V_1 + (1-\lambda)/V_2) + A(1-\theta)D_1}, \tag{IA.49}$$

$$a_0(\tilde{I}) = \bar{p}(\tilde{I}) - a_r(\tilde{I} + a_r\bar{q}/c) - a_s\bar{s}. \tag{IA.50}$$

By (9), the fee is $\Phi = (2A)^{-1}\ln(V_2/V_1)$. The seller therefore faces the same objective and feasible set as in the baseline. Applying Theorem 1 gives the same case boundaries, $\lambda^*, \Phi^*, \sigma_u^*$, and loadings; (IA.40) also gives the same δ^*. Only a_0 depends on $\tilde{I}$. □

IA.1.14 Proof of Lemma 7

1. Expected payoff. Fix $\tilde{I}$. By Lemma 6, the continuation loadings and $\delta^*(q) = a_r^* q/c$ are independent of the manager's private choice I. His full objective and conditional expected price are

$$W(I; \tilde{I}) = \mathbb{E}\left[qp - \frac{c}{2}\delta^*(q)^2\right] - \frac{c_I}{2}I^2, \tag{IA.51}$$

$$\mathbb{E}[p \mid I, q] = a_0(\tilde{I}) + a_r^* I + (a_r^*)^2 q/c + a_s^*\bar{s}. \tag{IA.52}$$

Using $\mathbb{E}[q^2] = \bar{q}^2 + \sigma_q^2$ gives

$$\mathbb{E}\left[qp - \frac{c}{2}\delta^*(q)^2\right] = \bar{q}\big(a_0(\tilde{I}) + a_r^* I + a_s^*\bar{s}\big) + \frac{(a_r^*)^2}{2c}(\bar{q}^2 + \sigma_q^2), \tag{IA.53}$$

$$\mathbb{E}[p] = a_0(\tilde{I}) + a_r^* I + (a_r^*)^2\bar{q}/c + a_s^*\bar{s}, \tag{IA.54}$$

$$\bar{q}\,\mathbb{E}[p] = \bar{q}\big(a_0(\tilde{I}) + a_r^* I + a_s^*\bar{s}\big) + (a_r^*)^2\bar{q}^2/c. \tag{IA.55}$$

Hence the two objectives differ by a constant independent of I:

$$W(I; \tilde{I}) = \bar{q}\,\mathbb{E}[p] - \frac{c_I}{2}I^2 + \frac{(a_r^*)^2}{2c}(\sigma_q^2 - \bar{q}^2). \tag{IA.56}$$

2. Optimal investment. The terms depending on I are $\bar{q}a_r^* I - c_I I^2/2$. Strict concavity gives the unique optimum $I^* = \bar{q}a_r^*/c_I$. Since a_r^* is independent of $\tilde{I}$, imposing the equilibrium belief $\tilde{I} = I^*$ leaves this optimum unchanged. □

IA.1.15 Proof of Proposition 6

Derivatives are taken along differentiable equilibrium branches within the respective regimes. By (21) and the maintained assumptions $\bar{q}, c_I > 0$,

$$\frac{dI^*}{dc} = \frac{\bar{q}}{c_I}\frac{da_r^*}{dc}. \tag{IA.57}$$

Use $z = -a_s/a_r$ and $D(z,\lambda) = 1/a_r$ from the baseline proofs. With $\kappa = \theta/(A\sigma_y^2) + (1-\theta)D_1$, the coefficient equations are

$$D(z,\lambda) = \frac{\rho}{\sigma_y} + \kappa z \frac{A\sigma_s^2\sigma_y^2 z + \rho\sigma_y\theta\lambda}{A\sigma_s^2\sigma_y^2 z + \rho\sigma_y\theta}, \tag{IA.58}$$

$$\rho^2 + \frac{\rho\theta\lambda^*}{A\sigma_y} z = 1 + (\sigma_u^*)^2 + \frac{\sigma_q^2}{c^2 D(z,\lambda^*)^2}. \tag{IA.59}$$

1. Unconstrained equilibrium. In the unconstrained equilibrium, Theorem 1 gives $\lambda^* = \min\{\overline{\lambda}, 1\}$ and

$$\mathrm{Var}(g_r) = \rho^2\left(1 + \frac{\theta\lambda^*}{A\sigma_s\sigma_y}\right), \tag{IA.60}$$

$$z = M^* = \rho/\sigma_s. \tag{IA.61}$$

Both λ^* and z are independent of c, so (IA.58) implies $da_r^*/dc = dI^*/dc = 0$.

2. Constrained equilibrium with full participation. Here $\sigma_u^* = \underline{\sigma}$, $\lambda^* = 1$, and $z = M(1)$ solves

$$F(z;c) \equiv \rho^2 + \frac{\rho\theta}{A\sigma_y} z - 1 - \underline{\sigma}^2 - \frac{\sigma_q^2}{c^2 D(z,1)^2} = 0. \tag{IA.62}$$

Since $D(z,1) = \rho/\sigma_y + \kappa z$, we have $F_z = \rho\theta/(A\sigma_y) + 2\sigma_q^2\kappa/(c^2 D^3) > 0$ and $F_c = 2\sigma_q^2/(c^3 D^2) > 0$.

Implicit differentiation within this regime yields

$$\frac{dz}{dc} = -\frac{F_c}{F_z} < 0, \tag{IA.63}$$

$$\frac{da_r^*}{dc} = -\frac{\kappa}{D(z,1)^2}\frac{dz}{dc} > 0. \tag{IA.64}$$

Thus $da_r^*/dc > 0$ under full participation.

3. Constrained equilibrium with partial participation. Here $\sigma_u^* = \underline{\sigma}$ and $0 < \lambda^* < 1$. Use the normalization and slope ratios in Steps 1–2 of Section IA.1.7, namely (IA.12)–(IA.15) and (IA.18). In the present comparative static, h is fixed and ν varies with c; suppress h in the arguments of $\mathcal{D}$ and $\mathcal{G}$. The same steps give $\zeta > 1$, $k\zeta > b$, $\widehat{\Pi}_\zeta < 0$, and $\mathcal{G}_k, \mathcal{G}_\zeta > 0$. Equations (IA.19) and (IA.21) establish $\mathcal{H}_k > 0 > \mathcal{T}_k$. These identities and slope signs require only $h \geqslant \bar{k}$, so they also apply when $\theta = 1$.

Since $\mathcal{G}_k > 0$, solve (IA.14) locally for $k = k(\zeta, \nu)$ and define

$$\mathcal{V}(\zeta,\nu) \equiv \widehat{\Pi}(k(\zeta,\nu),\zeta), \qquad k_\nu = \frac{1}{\mathcal{G}_k}, \quad k_\zeta = -\frac{\mathcal{G}_\zeta}{\mathcal{G}_k} = -\frac{1}{\mathcal{T}}. \tag{IA.65}$$

The seller's first-order condition is $\mathcal{V}_\zeta = \widehat{\Pi}_\zeta(1 - \mathcal{H}/\mathcal{T}) = 0$, or $\mathcal{H} = \mathcal{T}$. At such an optimum, (IA.65) and the two slope signs imply

$$\mathcal{V}_{\zeta\nu} = -\frac{\widehat{\Pi}_\zeta(\mathcal{H}_k - \mathcal{T}_k)}{\mathcal{T}\mathcal{G}_k} > 0. \tag{IA.66}$$

The second-order necessary condition gives $\mathcal{V}_{\zeta\zeta} \leqslant 0$. Along a differentiable equilibrium branch, differentiating $\mathcal{V}_\zeta = 0$ gives $\mathcal{V}_{\zeta\zeta}\, d\zeta^*/d\nu + \mathcal{V}_{\zeta\nu} = 0$. Thus (IA.66) rules out $\mathcal{V}_{\zeta\zeta} = 0$ and implies $d\zeta^*/d\nu > 0$.

Writing $\mathcal{D}^* = \mathcal{D}(k^*, \zeta^*)$, differentiation of (IA.14) now gives

$$\frac{d\mathcal{D}^*}{d\nu} = \frac{\mathcal{D}_k}{\mathcal{G}_k} + \left(\mathcal{D}_\zeta - \frac{\mathcal{D}_k\mathcal{G}_\zeta}{\mathcal{G}_k}\right)\frac{d\zeta^*}{d\nu} > 0, \tag{IA.67}$$

where all derivatives are evaluated at the optimum. The inequality follows from

$$\mathcal{D}_\zeta \mathcal{G}_k - \mathcal{D}_k \mathcal{G}_\zeta = \mathcal{D}^2(\zeta \mathcal{D}_\zeta - k\mathcal{D}_k) = \mathcal{D}^2 \frac{h\zeta^2(\zeta + 2\bar{k} - k)}{(\zeta + \bar{k})^2} > 0.$$

Finally, (IA.12), (IA.13), and (IA.67) yield

$$\frac{da_r^*}{dc} = \frac{2\sigma_y \nu}{\rho c (\mathcal{D}^*)^2} \frac{d\mathcal{D}^*}{d\nu} > 0.$$

Together with Step 2 and (IA.57), this proves $dI^*/dc > 0$ in the constrained equilibrium. □

IA.1.16 Proof of Lemma 8

Write $d = (1-\theta)D_2$, $\alpha = \rho/\sigma_y$, $\beta = \rho\theta/(A\sigma_y)$, $\kappa = \theta/(A\sigma_y^2) + (1-\theta)D_1$, and $S = \sigma_q^2/c^2$. Retain $z = -a_s/a_r$, the precisions r_1, r_2, and $V_i = \sigma_y^2(1-\rho^2 r_i)$ from the baseline proof. These definitions will also be used in the remaining mandate proofs.

1. Equilibrium and the relaxed seller problem. The manager still chooses $\delta = a_r q/c$. Adding mandate demand to the market-clearing calculation in the proof of Proposition 1 gives

$$\begin{aligned}\left[\theta\left(\frac{\lambda}{V_1} + \frac{(1-\lambda)(1-\rho\sigma_y r_2/a_r)}{V_2}\right) + A(1-\theta)D_1\right] dp \\ = \left(\theta\lambda\rho\sigma_y \frac{r_1}{V_1} + Ad\right) dg_r - A\, d\tilde{s}.\end{aligned} \tag{IA.68}$$

Matching coefficients and using $r_2^{-1} = r_1^{-1} + \sigma_s^2 z^2$ yields

$$\frac{A}{z} = \theta\lambda\rho\sigma_y \frac{r_1}{V_1} + Ad, \tag{IA.69}$$

$$r_1^{-1} = \rho^2 + \frac{\beta\lambda z}{1-dz}, \tag{IA.70}$$

$$r_2^{-1} - \rho^2 = \frac{\beta\lambda z}{1-dz} + \sigma_s^2 z^2, \tag{IA.71}$$

$$\Xi = \kappa - \alpha d = \frac{\theta}{A\sigma_y^2} + (1-\theta)(D_1 - \alpha D_2) > 0, \tag{IA.72}$$

$$\frac{1}{a_r} = \alpha + \Xi z \frac{\beta\lambda + \sigma_s^2 z(1-dz)}{\beta[1-(1-\lambda)dz] + \sigma_s^2 z(1-dz)}. \tag{IA.73}$$

The domain is $0 < z < 1/d$, with upper endpoint $+\infty$ if $d = 0$. To check monotonicity, let $T = \beta\lambda + \sigma_s^2 z(1 - dz)$ and $U = T + \beta(1 - \lambda)(1 - dz)$. Then $\partial_z(zT/U) = T/U + z\beta(1 - \lambda)[\sigma_s^2(1 - dz)^2 + d\beta\lambda]/U^2 > 0$ and $\partial_\lambda(T/U) = \beta(1 - dz)[\beta + \sigma_s^2 z(1 - dz)]/U^2 > 0$. Hence a_r decreases strictly in z and λ.

By Lemma 1, the seller maximizes $\lambda \ln(V_2/V_1)$, where

$$\frac{V_2}{V_1} = 1 + \frac{A\sigma_s^2\sigma_y^2}{\theta\lambda H^M(z, \lambda)}, \tag{IA.74}$$

$$H^M(z, \lambda) = \frac{\rho\sigma_y}{z(1 - dz)} + \frac{\sigma_s^2\sigma_y z}{\rho(1 - dz)} + \frac{\theta\lambda}{A(1 - dz)^2}. \tag{IA.75}$$

Ignoring the noise floor, H^M is strictly convex in z and diverges at both endpoints, so its minimizer $z^M(\lambda)$ is unique. Revenue vanishes uniformly as $\lambda \downarrow 0$ or z approaches an endpoint, using $H^M \geqslant 2\sigma_s\sigma_y$ and $\ln(1 + x) \leqslant x$; hence a relaxed maximum exists. If $d > 0$, $H^M_{z\lambda} = 2\theta d/[A(1 - dz)^3] > 0$, so z^M decreases with λ; its first-order condition gives $r_1 = 2dz/(\rho^2 - \sigma_s^2 z^2)$, so r_1 decreases as well. For $d = 0$, $z^M = \rho/\sigma_s$ and r_1 again decreases with λ.

At $z = z^M(\lambda)$, the envelope derivative satisfies

$$\frac{\frac{d}{d\lambda}[\lambda \ln(V_2/V_1)]}{1 - V_1/V_2} = \frac{\ln(V_2/V_1)}{1 - V_1/V_2} - 1 - \frac{1 - \rho^2 r_1}{1 + \sigma_s^2 z^2 r_1}. \tag{IA.76}$$

The minimized H^M increases with λ, so V_2/V_1 decreases; $\ln x/(1 - 1/x)$ increases for $x > 1$. The last fraction increases as z and r_1 decrease. Thus the right-hand side decreases strictly from $+\infty$, giving a unique relaxed optimum, either interior or at $\lambda = 1$. It is also the only pair satisfying the noise first-order condition and the necessary share condition. Every slack equilibrium must therefore coincide with this optimum.

2. The exact regime threshold. The binding frontier $z = M^M(\lambda, \sigma_q)$ is the unique root of

$$\rho^2 + \frac{\beta\lambda z}{1 - dz} = 1 + \underline{\sigma}^2 + Sa_r(z, \lambda)^2. \tag{IA.77}$$

The left side increases and the right side decreases with z; feasible choices are exactly $z \geqslant M^M$, with strict inequality equivalent to slack noise. The frontier decreases with λ and increases with

σ_q. Positive revenue is attainable for every $\lambda > 0$, so zero participation is never optimal, and the preceding boundary behavior ensures existence. Evaluating a_r, r_1 at the relaxed optimum, define

$$\sigma^{\dagger}_{q,M} = \frac{c}{a_r}\sqrt{\max\{r_1^{-1} - 1 - \underline{\sigma}^2, 0\}}. \tag{IA.78}$$

The required noise slack is $r_1^{-1} - 1 - \underline{\sigma}^2 - Sa_r^2$. It is positive exactly when $\sigma_q < \sigma^{\dagger}_{q,M}$; otherwise the relaxed target is at the floor or infeasible, and Step 1 excludes any other slack optimum. When $D_2 = 0$, the target is the baseline target, so (IA.78) equals (A.51) and $\sigma^{\dagger}_{q,M} = \sigma^{\dagger}_q$.

3. Full participation uniformly in D_2. Set $m = z/(1 - dz)$, so $z = m/(1 + dm)$. Equations (IA.73) and (IA.77) become

$$\frac{1}{a_r} = \frac{\alpha + \kappa m}{1 + dm} - \frac{\Xi(1 - \lambda)m}{(1 + dm)(1 + \lambda dm) + (\sigma_s^2/\beta)m}, \tag{IA.79}$$

$$\beta\lambda m = 1 + \underline{\sigma}^2 - \rho^2 + Sa_r^2. \tag{IA.80}$$

Since $1/a_r \leqslant \alpha + \kappa m$, the frontier satisfies $\beta m \geqslant S/(\alpha + \kappa m)^2$. Consequently, $m \to \infty$ uniformly over $\lambda \in (0, 1]$ and admissible D_2 as $S \to \infty$. In these coordinates, $H^M = \rho\sigma_y(1 + dm)^2/m + (\sigma_s^2\sigma_y/\rho)m + (\theta\lambda/A)(1 + dm)^2$, whose m-derivative is positive when $m > \rho/\sigma_s$. Thus the noise floor binds at every share for sufficiently large S, uniformly in D_2.

Let η be the ratio of the subtracted term in (IA.79) to the first term. Then

$$0 \leqslant \eta \leqslant \left[1 + \lambda dm + \frac{\sigma_s^2}{\beta}\frac{m}{1 + dm}\right]^{-1} = o(1) \tag{IA.81}$$

uniformly on the frontier. Indeed, along any sequence with $S \to \infty$, if $d \to 0$, then $m/(1 + dm) \to \infty$; if d stays bounded away from zero, $1/a_r \leqslant \alpha + \kappa/d$ and (IA.80) imply $\lambda dm \to \infty$. These cases cover subsequences of every admissible sequence.

For the subtracted term, the absolute value of its logarithmic m-derivative is at most one. The first term has logarithmic derivative at most $1/(1 + dm)$, so $0 < -ma_{r,m}/a_r \leqslant 1/(1 + dm) + o(1)$

uniformly. Implicit differentiation of (IA.80), using $a_{r,\lambda} < 0$, gives

$$-\frac{\lambda m_\lambda}{m} \geqslant \left(1 - \frac{2ma_{r,m}}{a_r}\right)^{-1} \geqslant \left(1 + \frac{2}{1+dm} + o(1)\right)^{-1}. \tag{IA.82}$$

For all sufficiently large S, this is at least $1/4$ everywhere and at least $3/4$ if $dm \geqslant 7$. Also $mH_m^M \geqslant (\sigma_s^2\sigma_y/2\rho)m + (2\theta\lambda/A)dm(1+dm)$. Along the frontier, $\lambda\, dH^M/d\lambda = (\theta\lambda/A)(1+dm)^2 + (\lambda m_\lambda/m)mH_m^M$. If $dm \leqslant 7$, this is at most $64\theta/A - (\sigma_s^2\sigma_y/8\rho)m < 0$; if $dm \geqslant 7$, it is negative because $3dm/2 > 1 + dm$. Hence H^M strictly decreases with the share along the frontier. The seller's objective there is

$$\Psi^M(\lambda;\sigma_q) = \lambda \ln\left[1 + \frac{A\sigma_s^2\sigma_y^2}{\theta\lambda H^M(M^M(\lambda,\sigma_q),\lambda)}\right]. \tag{IA.83}$$

For fixed H^M, this increases strictly in λ by $\ln(1+x) > x/(1+x)$; decreasing H^M raises it further. Thus $\lambda^* = 1$ uniquely. All bounds are uniform in D_2, giving a finite $\sigma_{q,M}^{\text{full}} \geqslant \sigma_{q,M}^{\dagger}$ independent of D_2. Increase this bound, if necessary, as specified in the next proof; the lemma continues to hold. □

IA.1.17 Proof of Proposition 7

Retain the preceding notation and let $D_2 \in [0, \sigma_y D_1/\rho)$ vary, fixing $\theta < 1$ and all other primitives except σ_q.

1. Score informativeness and the average score. For $\sigma_q \geqslant \sigma_{q,M}^{\text{full}}$, the preceding lemma gives $\lambda^* = 1$ and $\sigma_u^* = \underline{\sigma}$. Hence

$$r_1^{-1} = \rho^2 + \beta m, \tag{IA.84}$$

$$a_r = \frac{1+dm}{\alpha + \kappa m}, \tag{IA.85}$$

$$G(m, D_2) \equiv \rho^2 + \beta m - 1 - \underline{\sigma}^2 - S\frac{(1+dm)^2}{(\alpha+\kappa m)^2} = 0. \tag{IA.86}$$

Since $G(0, D_2) < 0$, $G(m, D_2) \to \infty$, and

$$G_m = \beta + \frac{2S(1+dm)\Xi}{(\alpha+\kappa m)^3} > 0, \tag{IA.87}$$

$$G_{D_2} = -\frac{2S(1-\theta)m(1+dm)}{(\alpha+\kappa m)^2} < 0, \tag{IA.88}$$

$$m'(D_2) = -G_{D_2}/G_m > 0, \tag{IA.89}$$

the root is unique and increases strictly in D_2. Now $\mathrm{Corr}(g_r, \tilde{g}) = (\rho^2 + \beta m)^{-1/2}$ and $\mathbb{E}[g_r] = (\bar{q}/c)\sqrt{(\rho^2 + \beta m - 1 - \underline{\sigma}^2)/S}$. Because $\bar{q} > 0$, score informativeness decreases strictly and the average score increases strictly.

2. The unique price-informativeness maximum. Price informativeness is $(\rho^2 + \beta m + \sigma_s^2 z^2)^{-1/2}$. Eliminating d using (IA.86) gives $z = m\sqrt{S}/[(\alpha+\kappa m)\sqrt{\rho^2 + \beta m - 1 - \underline{\sigma}^2}]$, so it equals $H(m)^{-1/2}$, where

$$H(m) = \rho^2 + \beta m + \frac{\sigma_s^2 S m^2}{(\rho^2 + \beta m - 1 - \underline{\sigma}^2)(\alpha+\kappa m)^2}. \tag{IA.90}$$

Differentiating this rational function yields

$$H'(m) = \beta - \frac{\sigma_s^2 S}{\beta\kappa^2 m^2}[1 + O(m^{-1})], \tag{IA.91}$$

$$H''(m) = \frac{2\sigma_s^2 S}{\beta\kappa^2 m^3}[1 + O(m^{-1})]. \tag{IA.92}$$

The remainder constants depend only on fixed primitives. Since $m(D_2) \geqslant m(0) \to \infty$, $H'' > 0$ throughout the equilibrium range for all sufficiently large S. At $D_2 = 0$, (IA.86) gives $m(0) \sim [S/(\beta\kappa^2)]^{1/3}$ and $H'(m(0)) = \beta - \sigma_s^2 m(0)[1 + o(1)] < 0$. For each fixed $D_2 > 0$, instead,

$$m(D_2) \sim \frac{d^2 S}{\beta\kappa^2}, \qquad H'(m(D_2)) \longrightarrow \beta > 0. \tag{IA.93}$$

Enlarge $\sigma_{q,M}^{\text{full}}$, independently of D_2, so that strict convexity, the negative derivative at zero, and a positive derivative at $D_2 = \sigma_y D_1/(2\rho)$ all hold whenever $\sigma_q \geqslant \sigma_{q,M}^{\text{full}}$. Continuity and strict convexity then give a unique $\widehat{D}_2 \in (0, \sigma_y D_1/\rho)$ with $H'(m(\widehat{D}_2)) = 0$. Since $m' > 0$, differentiating $H(m(D_2))^{-1/2}$ proves the claimed strict increase and decrease.

3. The turning point tends to zero. For every fixed admissible $D_2 > 0$, (IA.93) implies $H'(m(D_2)) > 0$ eventually. Uniqueness of the turning point gives $0 < \widehat{D}_2 < D_2$ eventually. As this holds for every such D_2, $\widehat{D}_2 \to 0$. □

IA.1.18 Mandate demand with partial participation

The following result extends the adverse effects of mandate demand in Proposition 7 to constrained equilibria with partial participation. Retain $d, \alpha, \beta, \kappa, S, \Xi, z$ from the proof of Lemma 8, and define

$$\Gamma \equiv 1 + \underline{\sigma}^2 - \rho^2 > 0, \qquad \tau \equiv r_1^{-1} - \rho^2 = \mathrm{Var}(g_r) - \rho^2. \tag{IA.94}$$

Proposition IA.2 (Small supply noise and partial participation)**.** *Maintain the mandate model with $0 < \theta < 1$ and $\sigma_q > 0$, fixing all primitives other than D_2 and σ_s. Fix a reference value $D_{2,0} \in (0, \sigma_y D_1/\rho)$, and write $d_0 = (1-\theta)D_{2,0}$. Suppose*

$$\beta > d_0\Gamma + \frac{4Sd_0^3}{(\kappa + \alpha d_0)^2}. \tag{IA.95}$$

There exist $\bar{\sigma}_s > 0$ and an open neighborhood of $D_{2,0}$ contained in $(0, \sigma_y D_1/\rho)$ such that, for every $0 < \sigma_s < \bar{\sigma}_s$ and every D_2 in this neighborhood, the equilibrium is unique, constrained, and has strictly partial participation: $\sigma_u^* = \underline{\sigma}$ and $0 < \lambda^* < 1$. Throughout this neighborhood,

$$\frac{d\ \mathrm{Corr}(g_r, \widetilde{g})}{dD_2} < 0, \qquad \frac{d\,\mathbb{E}[g_r]}{dD_2} > 0, \qquad \frac{d\ \mathrm{Corr}(p, \widetilde{g})}{dD_2} < 0. \tag{IA.96}$$

Moreover, $d\lambda^*/dD_2 > 0$ throughout the same neighborhood.

Proof. We establish the equilibrium regime and the strict comparative statics uniformly near d_0. Choose a compact neighborhood of d_0 inside $(0, (1-\theta)\sigma_y D_1/\rho)$ on which (IA.95) remains strict.

1. The limiting scaled problem. In the coordinates (IA.94), rearranging (IA.70) and (IA.73) gives

$$\lambda = \frac{\tau(1-dz)}{\beta z}, \qquad \frac{1}{a_r} = \alpha + \Xi\frac{\tau + \sigma_s^2 z^2}{\beta + d\tau + \sigma_s^2 z}, \qquad \tau \geqslant \Gamma + S a_r^2. \tag{IA.97}$$

The last inequality is the noise constraint, with equality exactly when $\sigma_u = \underline{\sigma}$. The share restriction is $\tau/(\beta + d\tau) \leqslant z < 1/d$. By (IA.74), the seller maximizes the normalized revenue

$$\Pi = \frac{\tau(1 - dz)}{\beta z} \ln\left(1 + \frac{\rho^2 \sigma_s^2 z^2}{\tau(\rho^2 + \tau + \sigma_s^2 z^2)}\right). \tag{IA.98}$$

For $\sigma_s > 0$, dividing by σ_s^2 preserves its maximizers. The resulting objective extends smoothly to $\sigma_s^2 = 0$, with limit

$$\frac{\Pi}{\sigma_s^2} \longrightarrow \frac{\rho^2 z(1 - dz)}{\beta(\rho^2 + \tau)}, \qquad a_r^0(\tau, d) = \frac{\beta + d\tau}{\alpha\beta + \kappa\tau}. \tag{IA.99}$$

Since $\partial_\tau a_r^0 = -\beta\Xi/(\alpha\beta + \kappa\tau)^2 < 0$, limiting feasibility is $\tau \geqslant \tau_0(d)$, where $\tau_0(d)$ is the unique root of

$$\tau_0 = \Gamma + S\left(\frac{\beta + d\tau_0}{\alpha\beta + \kappa\tau_0}\right)^2. \tag{IA.100}$$

The left side minus the right side is strictly increasing in τ_0. Evaluating it at β/d shows that (IA.95) is equivalent to $d\tau_0(d) < \beta$. Because (IA.99) strictly decreases in τ and $z(1 - dz)$ is uniquely maximized at $1/(2d)$, the limiting scaled problem has the unique global optimizer

$$\tau = \tau_0(d), \qquad z = \frac{1}{2d}, \qquad \lambda = \frac{d\tau_0(d)}{\beta} \in (0, 1). \tag{IA.101}$$

Thus the limiting optimum binds the noise floor and leaves the participation cap slack.

2. Persistence for positive supply noise. The inequality $\ln(1 + x) \leqslant x$ in (IA.98) implies

$$0 \leqslant \frac{\Pi}{\sigma_s^2} \leqslant \frac{\rho^2 z(1 - dz)}{\beta(\rho^2 + \tau)} \leqslant \frac{\rho^2}{4\beta d(\rho^2 + \tau)}.$$

Feasibility gives $\tau \geqslant \Gamma > 0$ and $z \geqslant \Gamma/(\beta + d\Gamma) > 0$. The bound therefore excludes $\tau \to \infty$ and $z \to 1/d$ for any sequence with normalized revenue bounded away from zero. Such revenue is attainable near (IA.101): the floor equation in (IA.97) has a smooth local root, since $\partial_\tau a_r < 0$, and the share cap remains slack. Consequently, all global maximizers lie in a common compact set and converge uniformly in d to (IA.101).

Near that limit, $\partial_\tau(\Pi/\sigma_s^2) < 0$, so every maximizer binds the noise floor exactly; lowering τ at

fixed z also relaxes the share cap. On this smooth floor, the limiting second derivative with respect to z is

$$-\frac{2\rho^2 d}{\beta(\rho^2+\tau_0(d))}<0,$$

because the limiting floor $\tau_0(d)$ is independent of z. Local strict concavity and the implicit function theorem therefore give a unique global optimizer for all sufficiently small positive σ_s, with C^1 convergence in d to the limiting policy. Compactness of the chosen neighborhood makes the supply-noise bound uniform. The equilibrium is consequently constrained and strictly partial throughout a neighborhood of d_0.

3. Comparative statics. Write $a_r^0(d)=a_r^0(\tau_0(d),d)$ and let primes denote total derivatives in d. Differentiating (IA.100) gives

$$(a_r^0)'=\frac{\tau_0(\alpha\beta+\kappa\tau_0)}{(\alpha\beta+\kappa\tau_0)^2+2Sa_r^0\beta\Xi}>0, \qquad \tau_0'=2Sa_r^0(a_r^0)'>0. \tag{IA.102}$$

For positive supply noise, the equilibrium mean score and correlations are

$$\mathbb{E}[g_r]=\frac{\bar{q}}{c}a_r^*, \qquad \mathrm{Corr}(g_r,\tilde{g})=(\rho^2+\tau^*)^{-1/2}, \qquad \mathrm{Corr}(p,\tilde{g})=(\rho^2+\tau^*+\sigma_s^2 z^{*2})^{-1/2}.$$

By (IA.102), their limiting d-derivatives are respectively $(\bar{q}/c)(a_r^0)'>0$ and $-\tau_0'/[2(\rho^2+\tau_0)^{3/2}]<0$ for both correlations. Also $(d\tau_0/\beta)'=(\tau_0+d\tau_0')/\beta>0$. The C^1 convergence from Step 2 preserves these strict signs uniformly after reducing the supply-noise bound if needed. Since $dd/dD_2=1-\theta>0$, they imply (IA.96) and the stated participation response. □

IA.1.19 Proof of Proposition 8

Fix σ_q and all other primitives except D_2, retaining the notation above. Because $\theta<1$, changes in d and D_2 have the same sign. By the proof of Lemma 8, every unconstrained equilibrium equals the unique relaxed optimum; first characterize this target, then check its feasibility.

1. The target score precision increases. The noise first-order condition $H_z^M = 0$ and (IA.70) give

$$\sigma_s^2 z^2 + \frac{2dz}{r_1} = \rho^2, \qquad d = \frac{r_1(\rho^2 - \sigma_s^2 z^2)}{2z}, \tag{IA.103}$$

so $0 < z \leqslant \rho/\sigma_s$. By (IA.76), the share derivative has the sign of

$$\frac{\ln(V_2/V_1)}{1 - V_1/V_2} - 1 - \frac{1 - \rho^2 r_1}{1 + \sigma_s^2 z^2 r_1}. \tag{IA.104}$$

This increases strictly in both r_1 and z: the ratio $V_2/V_1 = 1 + \rho^2\sigma_s^2 z^2 r_1^2/[(1 - \rho^2 r_1)(1 + \sigma_s^2 z^2 r_1)]$ increases, and the last fraction decreases. For fixed z, the expression ranges from -1 to $+\infty$ as r_1 ranges over $(0, 1/\rho^2)$; its zero thus defines a continuous, strictly decreasing precision as a function of z.

At the share cap, substituting (IA.103) into (IA.70) instead gives

$$\beta = \frac{(1 - \rho^2 r_1)(2 - \rho^2 r_1 + \sigma_s^2 z^2 r_1)}{2 r_1 z}. \tag{IA.105}$$

The right side decreases strictly in both r_1 and z on the stated domain, so its root also defines a continuous, strictly decreasing precision. The target precision is the larger of these two roots: $\lambda \leqslant 1$ requires precision at least as high as the cap root, and the share derivative is positive there exactly when the cap root exceeds the interior root. Thus target precision decreases strictly in z, including across switches of the share cap. The second expression in (IA.103) consequently decreases strictly in z along the target. Raising D_2 therefore lowers z and raises r_1. Hence both $\mathrm{Corr}(g_r, \tilde{g}) = \sqrt{r_1}$ and $\mathrm{Corr}(p, \tilde{g}) = (r_1^{-1} + \sigma_s^2 z^2)^{-1/2}$ increase strictly whenever the target is feasible.

2. The price loading and average score increase. The target satisfies

$$0 < r_1^{-1} - \rho^2 < \rho\sigma_s z. \tag{IA.106}$$

Indeed, at $r_1 = 1/(\rho^2 + \rho\sigma_s z)$, expression (IA.104) is negative by $\ln(1 + x) < x$. Its zero has larger

precision, and the target precision is at least that zero. Eliminating λ from (IA.73) gives

$$\frac{1}{a_r} = \alpha + \Xi \frac{r_1^{-1} - \rho^2 + \sigma_s^2 z^2}{\beta + \sigma_s^2 z + d(r_1^{-1} - \rho^2)}. \tag{IA.107}$$

The fraction increases in r_1^{-1} and decreases in d. At a point satisfying (IA.103), its partial z-derivative has the sign of $2\beta z + r_1[\rho^2\sigma_s^2 z^2 - (r_1^{-1} - \rho^2)^2] > 0$, by (IA.106). Substituting $d = r_1(\rho^2 - \sigma_s^2 z^2)/(2z)$ makes the fraction strictly increasing in both r_1^{-1} and z, since this substituted d is nonincreasing in r_1^{-1} and strictly decreasing in z. These signs hold on $0 < z \leqslant \rho/\sigma_s$, $\rho^2 < r_1^{-1} < \rho^2 + \rho\sigma_s z$. Comparing two targets by first lowering r_1^{-1} and then z stays in this domain, so the fraction decreases strictly with D_2. Since $\Xi > 0$ also decreases strictly, a_r increases strictly. Therefore $\mathbb{E}[g_r] = (\bar{q}/c)a_r$ increases strictly as well.

3. The unconstrained set is an initial interval. The target's required noise slack is $\sigma_u^2 - \underline{\sigma}^2 = r_1^{-1} - 1 - \underline{\sigma}^2 - Sa_r^2$. It is continuous and strictly decreasing in D_2, by Steps 1–2. Positive slack implements the unique unconstrained optimum; zero slack implements the same target at the floor; negative slack makes it infeasible, and the relaxed-optimum characterization excludes any other slack equilibrium. Thus the unconstrained set is an initial interval, possibly empty or the whole admissible interval. All three quantities increase strictly on this interval, and the equilibrium cannot return to it after becoming constrained. □

IA.1.20 Proof of Proposition 9

Use $z = -a_s/a_r$ and Π from the proof of Theorem 1, and set $\lambda^\dagger = \min\{\overline{\lambda}, 1\}$ and $M^* = \rho/\sigma_s$. Verification changes feasibility through $c(v)$ but leaves the revenue objective $\theta\Pi/(2A)$ unchanged.

1. Continuation choices and informativeness. At fixed v, the baseline proof applies with cost $c(v)$: feasible ratios form an upper interval, and increasing λ at fixed z preserves feasibility. Because $\Pi(\lambda, z)$ increases up to M^*, every continuation optimum has $z \geqslant M^*$. It also has $\lambda \geqslant \lambda^\dagger$. To establish the latter claim, set $k = \theta\lambda/(A\sigma_s\sigma_y)$, $t = k[k + \rho/(\sigma_s z) + \sigma_s z/\rho] \geqslant k(k+2)$, and $Q(t) = t(1+t)\ln(1+1/t) - t$. Differentiating (A.34) gives $\Pi_\lambda = [Q(t) - k^2]/[t(1+t)]$. Since $Q'(t) = (1+2t)\ln(1+1/t) - 2 > 0$, this derivative is positive whenever $k < \lambda_0$: the derivative of the

unconstrained objective in the baseline proof gives $Q(k(k+2)) > k^2$. Thus any $0 < \lambda < \lambda^\dagger$ can be improved by increasing participation at fixed z; zero participation is dominated by positive revenue.

By (A.25)–(A.26), the inverse squared score and price correlations are $\rho^2 + \rho\theta\lambda z/(A\sigma_y)$ and $\rho^2 + \rho\theta\lambda z/(A\sigma_y) + \sigma_s^2 z^2$, respectively. Both strictly increase in (λ, z), so neither correlation exceeds its value at $(\lambda^\dagger, M^*)$; equality in either requires both target values. Continuation revenue is continuous in v by the continuous baseline frontier and the uniform bound $0 \leqslant \Pi(\lambda, z) \leqslant \Pi(\lambda, M^*) \to 0$ as $\lambda \downarrow 0$. It is bounded above by $\theta\Pi(\lambda^\dagger, M^*)/(2A)$, while convexity gives $\psi(v) \geqslant \psi'(0)v \to \infty$. Hence a finite optimal verification level exists.

2. A feasible unconstrained target. If $M(\lambda^\dagger) \leqslant M^*$ at $v = 0$, the target remains feasible for every $v > 0$: increasing seller noise offsets the reduction in manipulation variance. This target globally maximizes gross revenue, so $\mathcal{R}(v) = \mathcal{R}(0)$ for all $v \geqslant 0$. Strictly increasing verification cost implies $v^* = 0$, and the equilibrium coincides with the baseline. This includes the regime boundary $M(\lambda^\dagger) = M^*$.

3. A strictly constrained baseline. Suppose $M(\lambda^\dagger) > M^*$ at $v = 0$. If $v^* = 0$, the baseline theorem gives a binding noise floor, and Step 1 gives strict informativeness inequalities because the target is infeasible. If $v^* > 0$, its continuation ratio cannot equal M^*. Indeed, hold (λ^*, σ_u^*) fixed and vary v. Equation (A.30) determines z smoothly because its left-hand side minus its right-hand side has a strictly positive derivative in z. At $z = M^*$, $\Pi_z = 0$, so the derivative of net profit along this feasible variation is $-\psi'(v^*) < 0$. A small reduction in verification would therefore raise profit. Hence $z > M^*$ at any positive optimum. The noise floor must bind, since otherwise reducing σ_u would lower z toward M^* and raise revenue. Step 1 then gives strict inequalities for both correlations. □

IA.1.21 Proof of Theorem 2

Write $z = -a_s/a_r$ for the ratio argument in (29), and use $M^* = \rho/\sigma_s$ and the revenue objective Π from the proof of Theorem 1.

1. Reduction and the design frontier. For fixed $(\lambda, \mathbf{w}) \in (0, 1] \times \mathcal{W}$, the manager's first-order conditions give $\delta_i^* = w_i a_r q/c_i$. Hence $g_r = \tilde{g} + \mathbf{w}^\top \mathbf{u} + a_r q/c(\mathbf{w})$, and the calculations in the proof

of Proposition 1 give the baseline pricing formulas with noise variance $\sigma_u^2(\mathbf{w})$ and cost $c(\mathbf{w})$. The induced ratio $z(\mathbf{w}, \lambda)$ is the unique positive root of

$$\rho^2 + \frac{\rho\theta\lambda}{A\sigma_y} z = 1 + \mathcal{N}(\mathbf{w}; \lambda, z). \tag{IA.108}$$

Indeed, the left-hand side strictly increases in z, whereas the right-hand side decreases by (A.28). At $z = 0$, the latter strictly exceeds ρ^2: $\sigma_q > 0$ and $1/c(\mathbf{w}) > 0$ ensure positive manipulation variance even if intrinsic noise vanishes. Thus the root exists and is unique. The seller's revenue is

$$\theta\lambda\Phi(\lambda, \mathbf{w}) = \frac{\theta}{2A}\Pi\big(\lambda, z(\mathbf{w}, \lambda)\big), \qquad \Pi(\lambda, z) = \lambda \ln(V_2/V_1). \tag{IA.109}$$

Compactness of $\mathcal{W}$ and continuity of $\mathcal{N}$ ensure that its minimum is attained. Replacing $\mathcal{N}$ in (IA.108) by its minimum gives a continuous, strictly increasing root equation in z, negative at zero and positive for sufficiently large z. Its unique root is

$$M_{\mathrm{sd}}(\lambda) = \min_{\mathbf{w}\in\mathcal{W}} z(\mathbf{w}, \lambda) = z\big(\mathbf{w}_{\max}(\lambda), \lambda\big). \tag{IA.110}$$

To verify the equality, evaluate the root equation for any $\mathbf{w}$ at $M_{\mathrm{sd}}(\lambda)$: its left-hand side minus its right-hand side is nonpositive, and equals zero for $\mathbf{w}_{\max}(\lambda)$. Strict monotonicity then gives the claim. The frontier is continuous by continuity and uniqueness of this root. It strictly decreases in λ, since increasing λ raises the left-hand side of (IA.108) and lowers a_r, hence weakly lowers every $\mathcal{N}(\mathbf{w}; \lambda, z)$, at fixed $z > 0$.

2. Optimal design at fixed participation. The proof of Theorem 1 shows that $\Pi(\lambda, z)$ strictly increases up to M^* and strictly decreases thereafter. Thus, writing $V(\lambda) = \max_{\mathbf{w}\in\mathcal{W}} \Pi(\lambda, z(\mathbf{w}, \lambda))$, the frontier property and the stated implementability assumption give

$$V(\lambda) = \Pi\big(\lambda, M_{\mathrm{sd}}(\lambda)\big), \qquad M_{\mathrm{sd}}(\lambda) > M^*, \tag{IA.111}$$

$$V(\lambda) = \Pi(\lambda, M^*), \qquad M_{\mathrm{sd}}(\lambda) \leqslant M^*. \tag{IA.112}$$

Equivalently,

$$V(\lambda) = \Pi\big(\lambda, \max\{M_{\text{sd}}(\lambda), M^*\}\big). \tag{IA.113}$$

The optimal weights minimize $\mathcal{N}$ in the first case and implement M^* in the second. Since $0 < V(\lambda) \leqslant \Pi(\lambda, M^*) \to 0$ as $\lambda \downarrow 0$, setting $V(0) = 0$ makes V continuous on $[0, 1]$ and ensures a positive maximum.

3. Participation and the three cases. By the proof of Theorem 1, $\Pi(\lambda, M^*)$ is strictly increasing up to $\overline{\lambda}$ and strictly decreasing thereafter, with maximizer $\lambda^\dagger = \min\{\overline{\lambda}, 1\}$ on $[0, 1]$. In the two unconstrained cases, $M_{\text{sd}}(\lambda^\dagger) \leqslant M^*$, so $V(\lambda) \leqslant \Pi(\lambda, M^*) \leqslant \Pi(\lambda^\dagger, M^*) = V(\lambda^\dagger)$. The implementing weights satisfy (A.6) or (A.7), respectively.

In the constrained case, $M_{\text{sd}}(\lambda^\dagger) > M^*$. No maximizer of V can have $M_{\text{sd}}(\lambda) < M^*$: continuity would make $V = \Pi(\lambda, M^*)$ locally, and strict unimodality would require $\lambda = \lambda^\dagger$, a contradiction (using one-sided changes at $\lambda = 1$). Every maximizer therefore lies on the design frontier. Since $\Pi(\lambda, M_{\text{sd}}(\lambda)) \leqslant V(\lambda)$ everywhere and equality holds at such a maximizer, maximizing the frontier objective gives exactly the same optimal value and a feasible optimal design $\mathbf{w}_{\max}(\lambda^*)$. Substitution of (A.34) yields (A.8)–(A.9). In all three cases, (9) gives the corresponding optimal fee. □

IA.1.22 Proof of Proposition A.2

1. Strict convexity. By Theorem 2, the conditional design problem holds $a_r^* = a_r(\lambda^*, M_{\text{sd}}(\lambda^*))$ fixed and minimizes (A.10). Let $\Delta_u = (\Sigma_u)_{11} + (\Sigma_u)_{22} - 2(\Sigma_u)_{12} = \text{Var}(u_1 - u_2) > 0$. Then $V_u'' = 2\Delta_u$, $m > 0$, and $m'' = 2(1/c_1 + 1/c_2) > 0$, so $L'' = 2\Delta_u + 2a_r^{*2}\sigma_q^2[(m')^2 + mm''] > 0$. Continuity and strict convexity give a unique minimizer on $[0, 1]$.

2. Location of the optimum. The benchmark definitions imply $V_u'(w) = 2\Delta_u(w - w_u)$ and $m'(w) = 2(1/c_1 + 1/c_2)(w - w_c)$. Hence $L' = V_u' + 2a_r^{*2}\sigma_q^2 mm'$ is negative below both benchmarks and positive above both. The unique minimizer therefore lies between w_u and w_c, proving (A.15). Both benchmarks are interior, so $L'(w^*) = 0$; its two terms also show that w^* is a convex combination of w_u and w_c. □