Internet Appendix for
"Would Order-By-Order Auctions Be Competitive?"

Thomas Ernst, Chester Spatt, and Jian Sun [1]

ABSTRACT

This Internet Appendix provides additional proofs supporting the main text.

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I. Omitted Proofs

Proof of Proposition 3

Since wholesalers cannot observe the realization of $\widetilde{c}_0$, they can condition their strategies only on distribution information about $\widetilde{c}_0$. We still focus on symmetric equilibria in this case, and conjecture that all wholesalers use the same bidding strategy

$$\widetilde{\beta}(y) = \widetilde{K}_0 + \tilde{K}_1 w,$$

with $\tilde{K}_1 > 0$. Similar to our baseline model, the wholesaler with lowest signal realization obtains the order in equilibrium. We follow the proof of Proposition 1, notably, the function $\tilde{v}(x,w) = \mathbb{E}\left[\tilde{\zeta}_i | \min_{-i} w_{-i} = x; w_i = w\right]$ is

$$\begin{aligned}
\tilde{v}(x,w) =& \mathbb{E}\left[\tilde{\zeta}_i | \min_{-i} w_{-i} = x; w_i = w\right] \\
=& p\mathbb{E}\left[\tilde{\zeta}_i | \min_{-i} w_{-i} = x; w_i = w; \exists j \neq i, w_j = y_j = x\right] \\
&+ (1-p)\,\mathbb{E}\left[\tilde{\zeta}_i | \min_{-i} w_{-i} = x; w_i = w; \nexists j \neq i, w_j = y_j = x\right] \\
=& p\left[c_0 + c_1\left(\frac{x + (N-2)\,p\frac{\frac{1}{2}+x}{2} + pw}{N}\right) + c_2 pw\right] \\
&+ (1-p)\left[c_0 + c_1\left(\frac{(N-2)\,p\frac{\frac{1}{2}+x}{2} + pw}{N}\right) + c_2 pw\right] \\
=& c_0 + c_1\left(\frac{px + (N-2)\,p\frac{\frac{1}{2}+x}{2} + pw}{N}\right) + c_2 pw,
\end{aligned}$$

which implies that $\tilde{v}(x,w) = v(x,w)$. The rest of the proof follows the proof of Proposition 1. So the equilibrium strategy is the same as that in the baseline model.

Similarly, note that $\mathbb{E}(\tilde{c}_0) = c_0$, the proof of welfare computation follows our proof of Lemma 1, and thus all welfare outcomes are the same as that in our baseline model.

Proof of Proposition 10

This proof replicates the proof of Proposition 1 and Proposition 2. Note that in this case, wholesalers observe the realization of $\widetilde{c}_0$, so we just need to change c_0 to $\widetilde{c}_0$

in Proposition 1 to obtain the equilibrium strategies in this extension. Regarding the welfare, since we consider the expected welfare of the investor, the investor's welfare will depend only on the distribution of $\widetilde{c}_0$, which is its expected value c_0.

Proof of Proposition 11

This proof replicates the proof of Proposition 7. Let δ_{c3} be the value that satisfies

$$\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} = \widetilde{K}_0^+ - \widetilde{K}_1^+ \cdot \frac{1}{2},$$

i.e.,

$$c_0 - \delta_{c1} + \frac{c_1}{4N}\left(N - 1 + \frac{2}{N}\right) + \frac{c_2}{2N} + \frac{N-1}{N}\left(\frac{c_1}{2}\frac{N+2}{N} + c_2\right)\frac{1}{2} \tag{IA.1}$$

$$= c_0 + \delta_{c1} + \frac{c_1\left(N + N_0 - 1 + \frac{2}{N+N_0}\right)}{4\left(N+N_0\right)} + \frac{c_2}{2\left(N+N_0\right)} - \frac{N+N_0-1}{N+N_0}\left(\frac{c_1}{2}\frac{N+N_0+2}{N+N_0} + c_2\right)\frac{1}{2}. \tag{IA.2}$$

Then when $\delta_c > \delta_{c3}$, $\left[\widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}, \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}\right] \cup \left[\widetilde{K}_0^+ - \widetilde{K}_1^+ \cdot \frac{1}{2}, \widetilde{K}_0^+ + \widetilde{K}_1^+ \cdot \frac{1}{2}\right] = \emptyset$. This implies that under the equilibrium conjectured, when $\widetilde{c}_0 = c_0 - \delta_c$, only wholesalers can obtain the order no matter what signals market participants observe.

We first verify that it is optimal for any wholesalers to choose $\widetilde{t}^-\left(y;\delta_c\right)$ if observing $c_0 - \delta_c$, given other market participants' strategies. When $\widetilde{c}_0 = c_0 - \delta_c$, then clearly $\widetilde{t}^-\left(y;\delta_c\right)$ is an equilibrium if we only have N wholesalers in the market, as suggested by Proposition 1. This is essentially the baseline model of order-by-order auctions with N bidders and unconditional expected inventory cost being $c_0 - \delta_c$. This means that it's optimal for any wholesalers to choose $\widetilde{t}^-\left(y;\delta_c\right)$ if there are only $N-1$ other wholesalers who also choose $\widetilde{t}^-\left(y;\delta_c\right)$ and no institutional traders in the market. Having N_0 institutional traders choose $\widetilde{t}^+\left(y;\delta_c\right)$ does not change this optimality, because given other wholesalers' choice $\widetilde{t}^-\left(y;\delta_c\right)$, the N_0 institutional traders will never obtain any order in any state when $\widetilde{c}_0 = c_0 - \delta_c$.

Then we verify that it's optimal for any wholesalers to choose $\widetilde{t}^+\left(y;\delta_c\right)$ if observing $c_0 + \delta_c$, given other market participants' strategies. Following our Proposition 1 in the baseline model of order-by-order auctions ($p = 1$), $\widetilde{t}^+\left(y;\delta_c\right)$ is an equilibrium with $N + N_0$ bidders and unconditional expected inventory cost being $c_0 + \delta_c$. So it's optimal for any wholesalers to choose $\widetilde{t}^+\left(y;\delta_c\right)$ if there are other $N + N_0 - 1$ bidders also choosing $\widetilde{t}^+\left(y;\delta_c\right)$.

Finally, we verify that it's optimal for any institutional trader i to choose $\widetilde{t}^+(y;\delta_c)$, given other market participants' strategies. Suppose the institutional trader i observes a cost shock y_i, then the institutional trader's utility is $V_i = \frac{1}{2}V_1\left(\widetilde{t}_i\right) + \frac{1}{2}V_2\left(\widetilde{t}_i\right)$, where V_1 (V_2) is institutional trader i's profit when the state is $c_0 - \delta_c$ ($c_0 + \delta_c$). It's clear that for any y_i, we have $\widetilde{t}^+(y_i;\delta_c) = \arg\max_{\widetilde{t}_i \in \left(\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}, \infty\right)} V_i$. This is because when $\widetilde{t}_i \in \left(\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}, \infty\right)$, $\left[\widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}, \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}\right] \cup \left(\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}, \infty\right) = \emptyset$, and thus we always have $V_1\left(\widetilde{t}_i\right) = 0$. And by definition $\widetilde{t}^+(y_i;\delta_c) = \arg\max_{\widetilde{t}_i \in \left(\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}, \infty\right)} V_2$. It is also clear that the wholesaler will never choose $\widetilde{t}_i < \widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}$, as any $\widetilde{t}_i < \widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}$ is dominated by $\widetilde{t}_i = \widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}$. Suppose that the institutional trader chooses $\widetilde{t}_i \in \left[\widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}, \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}\right]$. Note that $\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[(c_0 - \delta_c) - \frac{c_1+c_2}{2}\right] > 0$, and the upper bound of the profit in the case $c_0 - \delta_c$ is $\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[(c_0 - \delta_c) - \frac{c_1+c_2}{2}\right]$ because $\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}$ is the highest spread in $\widetilde{t}_i \in \left[\widetilde{K}_0^- - \widetilde{K}_1^- \cdot \frac{1}{2}, \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2}\right]$ and $(c_0 - \delta_c) - \frac{c_1+c_2}{2}$ is the lowest inventory cost. Besides, in the case $c_0 + \delta_c$, the institutional trader i will obtain the order with probability one. And an upper bound of the profit is $\widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[(c_0 + \delta_c) - \frac{c_1+c_2}{2}\right]$. Then $V_1 \le \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[(c_0 - \delta_c) - \frac{c_1+c_2}{2}\right]$ and $V_2 \le \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[(c_0 + \delta_c) - \frac{c_1+c_2}{2}\right]$. This implies $V_i \le \frac{1}{2}V_1 + \frac{1}{2}V_2 \le \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[c_0 - \frac{c_1+c_2}{2}\right]$. Then

$$V_i < 0 \Longleftarrow \widetilde{K}_0^- + \widetilde{K}_1^- \cdot \frac{1}{2} - \left[c_0 - \frac{c_1+c_2}{2}\right] < 0 \Longleftrightarrow \delta_c > \delta_{c4},$$

where

$$\delta_{c4} = \frac{c_1}{4N}\left(N - 1 + \frac{2}{N}\right) + \frac{c_2}{2N} + \widetilde{K}_1^- \cdot \frac{1}{2} + \frac{c_1+c_2}{2}. \tag{IA.3}$$

Let $\underset{\sim}{\delta} = \max\{\delta_{c3}, \delta_{c4}\}$ where δ_{c3} is defined by (IA.2) and δ_{c4} is defined by (IA.3). Then when $\delta_c > \underset{\sim}{\delta}$, we have $\widetilde{t}^+(y_i;\delta_c) = \widetilde{K}_0^+ + \widetilde{K}_1^+ y_i = \arg\max_{\widetilde{t}_i \in (-\infty,\infty)} V_i$, which implies that it is optimal for any institutional trader i to choose $\widetilde{t}^+(y;\delta_c)$, given other market participants' strategies.

Proof of Lemma 3

First, from Assumption 3, it's clear that $\{y_i\}_i$ are i.i.d. To see its distribution, first, note that y_i takes values in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. For any $Y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, let $\underline{Y} = -\frac{1}{2} + \left\lfloor \frac{Y - \left(-\frac{1}{2}\right)}{\frac{1}{2A+1}} \right\rfloor \cdot \frac{1}{2A+1}$.

Then

$$
\begin{aligned}
\text{Prob}\left(y_i \leq Y\right) &= \text{Prob}\left(y_i \leq \underline{Y}\right) + \text{Prob}\left(\underline{Y} < y_i \leq Y\right) \\
&= \left\lfloor \frac{Y - \left(-\frac{1}{2}\right)}{\frac{1}{2A+1}} \right\rfloor \cdot \frac{1}{2A+1} + \frac{1}{2A+1} \frac{Y - \underline{Y}}{\frac{1}{2A+1}} \\
&= \frac{\underline{Y} - \left(-\frac{1}{2}\right)}{\frac{1}{2A+1}} \cdot \frac{1}{2A+1} + \frac{1}{2A+1} \frac{Y - \underline{Y}}{\frac{1}{2A+1}} \\
&= Y - \left(-\frac{1}{2}\right).
\end{aligned}
$$

So y_i follows a uniform distribution $U\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof of Proposition 13

We conjecture that under broker's routing, there exists a symmetric equilibrium, in which all wholesalers choose the same strategy

$$
s\left(w; A\right) = A_0 + A_1 w.
$$

Consider any $i \in \{1, 2, 3...N\}$, and $(x, w) \in \left[-\frac{1}{2}\frac{1}{2A+1}, \frac{1}{2}\frac{1}{2A+1}\right] \times \left[-\frac{1}{2}\frac{1}{2A+1}, \frac{1}{2}\frac{1}{2A+1}\right]$, let $H\left(x|w\right) = \text{Prob}\left(\min_{-i} w_{-i} \leq x | w_i = w\right)$ and $h\left(x|w\right) = \frac{dH(x|w)}{dx}$. It is easy to show that $H\left(x|w\right) = 1 - \left(\frac{1}{2} - \left(2A+1\right)x\right)^{N-1}$, and $h\left(x|w\right) = \left(2A+1\right)\left(N-1\right)\left(\frac{1}{2} - \left(2A+1\right)x\right)^{N-2}$. Note that

$$
\begin{aligned}
\zeta_i &= c_0 + c_1 \frac{1}{N} \sum_{j=1}^{N} y_j + c_2 y_i \\
&= c_0 + c_1 \frac{1}{N} \sum_{j=1}^{N} \left(w_j + l_j\right) + c_2 \left(w_i + l_i\right).
\end{aligned}
$$

Let

$$\begin{aligned}
v(x,w) &= \mathbb{E}\left(\zeta_i | \min_{-i} w_{-i} = x, w_i = w\right) \\
&= \mathbb{E}\left(c_0 + c_1 \frac{1}{N}\sum_{j=1}^{N}(w_j + l_j) + c_2(w_i + l_i) \,|\, \min_{-i} w_{-i} = x, w_i = w\right) \\
&= \mathbb{E}\left(c_0 + c_1 \frac{1}{N}\sum_{j=1}^{N} w_j + c_2 w_i | \min_{-i} w_{-i} = x, w_i = w\right) \\
&= c_0 + c_1 \frac{1}{N}\left[w + x + (N-2)\frac{x + \frac{1}{2}\frac{1}{2A+1}}{2}\right] + c_2 w.
\end{aligned}$$

We focus on symmetric equilibria. Suppose all of wholesaler i's opponents use a continuous, increasing bid strategy $B(w) = A_0 + A_1 w$ at time 0. When wholesaler i observes signal $w \in \left[-\frac{1}{2}\frac{1}{2A+1}, \frac{1}{2}\frac{1}{2A+1}\right]$ and reports signal $z \in \left[-\frac{1}{2}\frac{1}{2A+1}, \frac{1}{2}\frac{1}{2A+1}\right]$, its expected profit is

$$\begin{aligned}
U_i(z,w) &= \text{Prob}\left(z \leq \min_{-i} w_{-i} | w_i = w\right)\left[B(z) - \mathbb{E}\left(\zeta_i | z \leq \min_{-i} w_{-i}, w_i = w\right)\right] \\
&= [1 - H(z|w)]\left[B(z) - \frac{1}{1 - H(z|w)}\int_z^{\frac{1}{2}\frac{1}{2A+1}} h(x|w)\, v(x,w)\, dx\right] \\
&= [1 - H(z|w)]\, B(z) - \int_z^{\frac{1}{2}\frac{1}{2A+1}} h(x|w)\, v(x,w)\, dx.
\end{aligned}$$

Wholesaler i's marginal incentive is characterize by

$$\begin{aligned}
&\frac{\partial U_i(z,w)}{\partial z} \\
&= -h(z|w)\, B(z) + (1 - H(z|w))\, B'(z) + h(z|w)\, v(z,w) \\
&= h(z|w)\left[-B(z) + \frac{1 - H(z|w)}{h(z|w)} B'(z) + v(z,w)\right] \\
&= h(z|w)\left[\begin{array}{c} -A_0 - A_1 w + \frac{\left(\frac{1}{2} - (2A+1)z\right)^{N-1}}{(2A+1)(N-1)\left(\frac{1}{2} - (2A+1)z\right)^{N-2}} A_1 + \\ c_0 + c_1 \frac{1}{N}\left[w + z + (N-2)\frac{z + \frac{1}{2}\frac{1}{2A+1}}{2}\right] + c_2 w \end{array}\right] \\
&= h(z|w)\left[\begin{array}{c} -A_0 - A_1 w + \frac{\left(\frac{1}{2(2A+1)} - z\right)}{(N-1)} A_1 + c_0 \\ + c_1 \frac{1}{N}\left[w + z + (N-2)\frac{z + \frac{1}{2}\frac{1}{2A+1}}{2}\right] + c_2 w \end{array}\right].
\end{aligned}$$

We conjecture that in equilibrium we have $\left.\frac{\partial U_i(z,w)}{\partial z}\right|_{z=w}=0$, this implies

$$-A_0-A_1w+\frac{\frac{1}{2}\frac{1}{2A+1}-w}{(N-1)}A_1+c_0+c_1\frac{1}{N}\left[w+w+(N-2)\frac{w+\frac{1}{2}\frac{1}{2A+1}}{2}\right]+c_2w=0.$$

The above condition holds for any w, then

$$-A_0+\frac{\frac{1}{2}\frac{1}{2A+1}}{(N-1)}A_1+c_0+c_1\frac{1}{N}\left[(N-2)\frac{\frac{1}{2}\frac{1}{2A+1}}{2}\right]=0,$$

and

$$-A_1-\frac{1}{(N-1)}A_1+c_1\frac{1}{N}\left[2+(N-2)\frac{1}{2}\right]+c_2=0.$$

The solution of the above two conditions are

$$A_0=c_0+c_1\frac{1}{2N(2A+1)}\left(\frac{1}{N}+\frac{N}{2}-\frac{1}{2}\right)+c_2\frac{1}{2N(2A+1)},$$

$$A_1=\frac{N-1}{N}\left(\frac{c_1}{2}\frac{N+2}{N}+c_2\right).$$

We also need to verify that $\left.\frac{\partial U_i(z,w)}{\partial z}\right|_{z=w}=0$ is a sufficient condition for optimization. This is clear as $\frac{\partial U_i(z,w)}{\partial z}$ is a linear function of z.

Proof of Proposition 14

First, we introduce the random variable $r=\min_i w_i\in\left[-\frac{1}{2}\frac{1}{2A+1},\frac{1}{2}\frac{1}{2A+1}\right]$. Its CDF is $D(r)=1-\left(\frac{1}{2}-(2A+1)r\right)^N$ and PDF is $d(r)=N(2A+1)\left(\frac{1}{2}-(2A+1)r\right)^{N-1}$. Then the investor's welfare is

$$
\begin{aligned}
W_I(A) &= -\mathbb{E}\left(A_0 + A_1 r\right) \\
&= -\int_{-\frac{1}{2(2A+1)}}^{\frac{1}{2(2A+1)}} \left(A_0 + A_1 r\right) N \left(2A+1\right) \left(\frac{1}{2} - \left(2A+1\right) r\right)^{N-1} dr \\
&= -\int_{-\frac{1}{2(2A+1)}}^{\frac{1}{2(2A+1)}} \left(A_0 + \frac{A_1}{2A+1} \left(2A+1\right) r\right) N \left(\frac{1}{2} - \left(2A+1\right) r\right)^{N-1} d\left[\left(2A+1\right) r\right] \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(A_0 + \frac{A_1}{2A+1} x\right) N \left(\frac{1}{2} - x\right)^{N-1} dx \\
&= -A_0 N \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - x\right)^{N-1} dx\right] - \frac{A_1}{2A+1} N \int_{-\frac{1}{2}}^{\frac{1}{2}} x \left(\frac{1}{2} - x\right)^{N-1} dx \\
&= -A_0 + \frac{A_1 \left(N-1\right)}{2\left(2A+1\right)\left(N+1\right)} \\
&= -c_0 - c_1 \frac{1}{\left(2A+1\right) N \left(N+1\right)} + c_2 \frac{\left(N-3\right)}{2\left(2A+1\right)\left(N+1\right)}.
\end{aligned}
$$

Total welfare is

$$
\begin{aligned}
& W_{total}(A) \\
=& \mathbb{E}\left[\mathbb{E}\left(-c_0 - c_1 \frac{1}{N} \sum_{i=1}^{N} y_j - c_2 y_i | w_i = \min_j w_j = r\right)\right] \\
=& \mathbb{E}\left[\mathbb{E}\left(-c_0 - c_1 \frac{1}{N} \sum_{i=1}^{N} w_j - c_2 r | \min_j w_j = r\right)\right] \\
=& \mathbb{E}\left[-c_0 - c_1 \frac{1}{N}\left[r + \left(N-1\right) \frac{r + \frac{1}{2}\frac{1}{2A+1}}{2}\right] - c_2 r\right] \\
=& -\int_{-\frac{1}{2(2A+1)}}^{\frac{1}{2(2A+1)}} \left(c_0 + c_1 \frac{1}{N}\left[r + \left(N-1\right) \frac{r + \frac{1}{2}\frac{1}{2A+1}}{2}\right] + c_2 r\right) N \left(2A+1\right) \left(\frac{1}{2} - \left(2A+1\right) r\right)^{N-1} dr \\
=& -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(c_0 + \frac{c_1}{\left(2A+1\right)} \frac{1}{N}\left[x + \left(N-1\right) \frac{x + \frac{1}{2}}{2}\right] + \frac{c_2}{\left(2A+1\right)} x\right) N \left(\frac{1}{2} - x\right)^{N-1} dx \\
=& -c_0 + \frac{N-1}{2\left(2A+1\right)\left(N+1\right)} c_2.
\end{aligned}
$$

The expected welfare of wholesalers is

$$\begin{aligned}
W_W(A) &= W_{total} - W_I \\
&= +\frac{N-1}{2(2A+1)(N+1)}c_2 - \left[-c_1\frac{1}{(2A+1)N(N+1)} + c_2\frac{(N-3)}{2(2A+1)(N+1)}\right] \\
&= \frac{\frac{c_1}{N}+c_2}{(N+1)(2A+1)}.
\end{aligned}$$

When $A=0$, the welfare outcomes are equivalent to the welfare outcomes under order-by-order auctions. First, the total welfare is a decreasing function of A, so

$$W_{total}(A) < W(0)$$

for any $A>0$. This implies that the total welfare is always higher under order-by-order auctions. Second, the expected welfare of wholesalers is a decreasing function of A, so

$$W_W(A) < W_W(0).$$

This implies that wholesalers' welfare is also higher under order-by-order auctions. Regarding the investor's welfare,

$$\begin{aligned}
& W_I(A) < W_I(0) \\
\Longleftrightarrow & -c_1\frac{1}{(2A+1)N(N+1)} + c_2\frac{(N-3)}{2(2A+1)(N+1)} < -c_1\frac{1}{N(N+1)} + c_2\frac{(N-3)}{2(N+1)} \\
\Longleftrightarrow & \frac{1}{2A+1}\left[-c_1\frac{1}{N(N+1)} + c_2\frac{(N-3)}{2(N+1)}\right] < -c_1\frac{1}{N(N+1)} + c_2\frac{(N-3)}{2(N+1)} \\
\Longleftrightarrow & -c_1\frac{1}{N(N+1)} + c_2\frac{(N-3)}{2(N+1)} > 0 \\
\Longleftrightarrow & \frac{N(N-3)}{2} > \frac{c_1}{c_2}.
\end{aligned}$$

Proof of Proposition 15

Consider any $i \in \{1,2...N\}$ and $(x,w) \in \left[-\frac{1}{2},\frac{1}{2}\right] \times \left[-\frac{1}{2},\frac{1}{2}\right]$, let $G(x|w) = \text{Prob}\left[\min_{-i} w_{-i} \le x | w_i = w\right]$ and $g(x|w) = \frac{dG(x|w)}{dx}$ It is easy to show that $G(x|w) =$

$1-\left(\frac{1}{2}-x\right)^{N-1}$, and $g\left(x|w\right)=\left(N-1\right)\left(\frac{1}{2}-x\right)^{N-2}$. Let

$$\begin{aligned}
v\left(x,w\right)=&\mathbb{E}\left[\zeta_i|\min_{-i}w_{-i}=x;w_i=w\right]\\
=&p\mathbb{E}\left[\zeta_i|\min_{-i}w_{-i}=x;w_i=w;\exists j\neq i,w_j=y_j=x\right]\\
&+\left(1-p\right)\mathbb{E}\left[\zeta_i|\min_{-i}w_{-i}=x;w_i=w;\nexists j\neq i,w_j=y_j=x\right]\\
=&p\left[\mathbb{E}\left(c_0\right)+\mathbb{E}\left(c_1\right)\left(\frac{x+\left(N-2\right)p\frac{\frac{1}{2}+x}{2}+pw}{N}\right)+\mathbb{E}\left(c_2\right)pw\right]\\
&+\left(1-p\right)\left[\mathbb{E}\left(c_0\right)+\mathbb{E}\left(c_1\right)\left(\frac{\left(N-2\right)p\frac{\frac{1}{2}+x}{2}+pw}{N}\right)+c_2pw\right]\\
=&\bar{c}_0+\bar{c}_1\left(\frac{px+\left(N-2\right)p\frac{\frac{1}{2}+x}{2}+pw}{N}\right)+\bar{c}_2pw.
\end{aligned}$$

We focus on symmetric equilibria. Suppose all of wholesaler i's opponents use a continuous, increasing bid strategy $\bar{B}\left(w\right)=\bar{K}_0+\bar{K}_1w$ at time 0. When wholesaler i observes signal w and reports signal z, its expected profit is

$$\begin{aligned}
U_i\left(z,w\right)&=\text{Prob}\left(z\leq\min_{-i}w_{-i}|w_i=w\right)\left[\bar{B}\left(z\right)-\mathbb{E}\left(\zeta_i|z\leq\min_{-i}w_{-i},w_i=w\right)\right]\\
&=\left[1-G\left(z|w\right)\right]\left[\bar{B}\left(z\right)-\frac{1}{1-G\left(z|w\right)}\int_z^1 g\left(x|w\right)v\left(x,w\right)dx\right]\\
&=\left[1-G\left(z|w\right)\right]\bar{B}\left(z\right)-\int_z^1 g\left(x|w\right)v\left(x,w\right)dx.
\end{aligned}$$

Wholesaler i's marginal incentive is characterized by

$$\begin{aligned}
&\frac{\partial U_i\left(z,w\right)}{\partial z}\\
=&-g\left(z|w\right)\bar{B}\left(z\right)+\left(1-G\left(z|w\right)\right)\bar{B}'\left(z\right)+g\left(z|w\right)v\left(z|w\right)\\
=&g\left(z|w\right)\left[-\bar{B}\left(z\right)+\left(\frac{1-G\left(z|w\right)}{g\left(z|w\right)}\right)\bar{B}'\left(z\right)+v\left(z|w\right)\right]\\
=&g\left(z|w\right)\left[-\bar{K}_0-\bar{K}_1z+\frac{\frac{1}{2}-z}{N-1}\bar{K}_1+\bar{c}_0+\bar{c}_1\left(\frac{pz+\left(N-2\right)p\frac{\frac{1}{2}+z}{2}+pw}{N}\right)+c_2pw\right].
\end{aligned}$$

We conjecture that in equilibrium we have

$$\left.\frac{\partial U_i(z,w)}{\partial z}\right|_{z=w}=0. \tag{IA.4}$$

This implies

$$-\left(\bar{K}_0+\bar{K}_1 w\right)+\frac{\frac{1}{2}-w}{N-1}\bar{K}_1+\bar{c}_0+\bar{c}_1\left(\frac{pw+(N-2)\,p\frac{\frac{1}{2}+w}{2}+pw}{N}\right)+\bar{c}_2 pw=0.$$

Since the above condition holds for all w, then $\bar{K}_0$,$\bar{K}_1$ are solved by $-\bar{K}_0+\frac{\frac{1}{2}\bar{K}_1}{N-1}+\bar{c}_0+\bar{c}_1\frac{N-2}{4N}p=0$, and $-\bar{K}_1-\frac{\bar{K}_1}{N-1}+\bar{c}_2 p+\frac{2\bar{c}_1 p}{N}+\frac{\bar{c}_1(N-2)p}{2N}=0$. We then get

$$\bar{K}_1=\frac{N-1}{N}p\left(\bar{c}_1\left(\frac{1}{2}+\frac{1}{N}\right)+\bar{c}_2\right), \tag{IA.5}$$

$$\bar{K}_0=\bar{c}_0+\frac{p}{2N}\left[\bar{c}_1\left(\frac{1}{N}+\frac{N}{2}-\frac{1}{2}\right)+\bar{c}_2\right]. \tag{IA.6}$$

We also need to verify that condition (IA.4) is a sufficient condition for optimization. Note that $g(z|w)>0$ and

$$-\bar{K}_0-\bar{K}_1 z+\frac{\frac{1}{2}-z}{N-1}\bar{K}_1+\bar{c}_0+\bar{c}_1\left(\frac{pz+(N-2)\,p\frac{\frac{1}{2}+z}{2}+pw}{N}\right)+c_2 pw$$

is linear in z, then it is clear that with (IA.5) and (IA.6), we must have that for all w, $\frac{\partial U_i(z,w)}{\partial z}<0 \Longleftrightarrow z>w$, confirming that (IA.4) is a sufficient condition for optimization.

Proof of Lemma 4

We introduce the random variable $r=\min_i w_i\in\left[-\frac{1}{2},\frac{1}{2}\right]$ and its CDF $H(r)=1-\left(\frac{1}{2}-r\right)^N$ and PDF $h(r)=N\left(\frac{1}{2}-r\right)^{N-1}$. First, we know that in our baseline model of broker's routing, total welfare is

$$W^{BR}_{total}=-\left(c_0-p\frac{N-1}{N+1}\frac{c_2}{2}\right).$$

Total welfare depends only on inventory allocation but not equilibrium spread, as the equilibrium spread is just a transfer between market makers and investors. In our extension of heterogeneous stocks, it is still the wholesaler with lowest cost shock y that obtains the order, so the order allocation is the same as that in our baseline model for any stocks (c_0, c_1, c_2). Then the total welfare is this extension satisfies

$$\bar{W}^{BR}_{heter,total} = W^{BR}_{total} = -\left(c_0 - p\frac{N-1}{N+1}\frac{c_2}{2}\right).$$

Since the equilibrium bidding strategy is $T(w) = \bar{K}_0 + \bar{K}_1 w$, the investor's welfare is

$$\begin{aligned}
\bar{W}^{BR}_{heter,I} &= -\mathbb{E}\left[\bar{K}_0 + \bar{K}_1 r | \min_i w_i = r\right] \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\bar{K}_0 + \bar{K}_1 r\right) N \left(\frac{1}{2} - r\right)^{N-1} dr \\
&= -\left[\bar{c}_0 + p\frac{2\bar{c}_1 - (N-3)\,N\bar{c}_2}{2N\,(1+N)}\right].
\end{aligned}$$

By $\bar{W}^{BR}_{heter,W} = \bar{W}^{BR}_{heter,total} - \bar{W}^{BR}_{heter,I}$, we know

$$\begin{aligned}
\bar{W}^{BR}_{heter,W} &= -\left(c_0 - p\frac{N-1}{N+1}\frac{c_2}{2}\right) + \left[\bar{c}_0 + p\frac{2\bar{c}_1 - (N-3)\,N\bar{c}_2}{2N\,(1+N)}\right] \\
&= (\bar{c}_0 - c_0) + \frac{p}{2\,(N+1)}\left[(N-1)\,c_2 - (N-3)\,\bar{c}_2\right] + p\frac{\bar{c}_1}{N\,(1+N)}.
\end{aligned}$$