A Dynamic Delegated Investment Model of SPACs*

Dan Luo[†] Jian Sun[‡]

September 23, 2021

Abstract

We study SPACs in a continuous-time delegated investment model. Our model is built upon three unique features of SPACs: the sponsor and the investor are only partially aligned, a SPAC has a short time horizon, and the investor has the final control over investment approval. Due to the misalignment in incentives, the sponsor has an increasing incentive to propose unprofitable projects to the investor; in response, the investor exerts more stringent screening based on her information. Although the screening helps curb the sponsor's moral hazard, it also dampens the disciplining effect of partial alignment in incentives. When the investor's information is sufficiently noisy, the second effect dominates, so giving the investor the control over investment approval reduces everyone's welfare. This adverse effect is more pronounced if entrepreneurs' strategic choices of SPAC or the sponsor's strategic choice of effort are considered. We find that introducing public assessment and making the investor's control right contingent on it may benefit both parties. We also explore whether a SPAC should be allowed to continue after the current project is disapproved by the investor.

Keywords: SPAC, delegated investment, dynamic delegation, moral hazard

JEL Codes: D82, D86, G23

*We thank Jeffrey Zwiebel, Steve Grenadier for helpful comments.
[†]Graduate School of Business, Stanford University; dluo0123@stanford.edu.
[‡]Sloan School of Management, Massachusetts Institute of Technology; jiansun@mit.edu.

1 Introduction

The past year (2020) has witnessed a remarkable rise of the special purpose acquisition company (SPAC). A SPAC is a company with no operations that offers securities for cash and places substantially all the offering proceeds into a trust or escrow account for future use in the acquisition of one or more private operating companies[1]. According to the calculation of Gahng et al. (2021), in 2020, "a total of 248 SPAC IPOs raised $75.3 billion" while 165 operating company IPOs raised $61.9 billion. As SPAC appears to be a major way that private companies raise money and go public, there emerges a heated debate among practitioners and the academia over the consequences and the future of SPAC. Proponents praise SPACs for their agility and flexibility to accommodate financing needs better than traditional ways. Opponents, citing the poor returns in the long history of blank-check companies, denounce SPACs as "bubbles" and "scams"[2]. Meanwhile, it is worth noting that SPAC is still a rapidly evolving industry. Practitioners are consistently experimenting with different practice, while the regulator is also pondering over how to ensure healthy growth of the industry. Therefore, understanding the economic mechanism of the current SPAC practice not only facilitates proper use of SPAC but also guides potential improvement of it.

Although there has been considerable empirical literature evaluating the performance of SPAC, we see little theoretical analysis on its underlying economic mechanisms. This paper intends to narrow the gap. We regard SPAC as a kind of delegated investment vehicles and focus on the strategic interaction between the SPAC sponsor and the SPAC investor. In theory, SPACs merit a special analysis because it differs from other common delegated investment vehicles such as private equity, hedge funds, and mutual funds in several aspects. First, the sponsor's payoff is not strongly linked to the actual performance of the investment, so the sponsor may prefer to do a deal unfavorable to the investor. Such systematic misalignment between the sponsor and the investor is minor in other cases. Second, SPACs feature a relatively short horizon. Typically, a SPAC will be liquidated absent a successful merger within 24 months while it is 10 years for private equity funds. Third, a SPAC leaves the final decision over investment to the investor, so the investor is heavily involved in the SPAC's operation.

Based on these unique features, we build a finite-horizon continuous-time model of the dynamic SPAC game with one sponsor and one representative investor. In the SPAC game, the spon-

[1] SEC website.

[2] "I have never found any blank-check investment vehicle attractive. No matter what the reputation or what the sponsor might be. . . . They are the ultimate in terms of lack of transparency."—Arthur Levitt, former SEC Chairman.

sor receives projects stochastically over time and decides whether to propose one to the investor in the form of tender offer. When a project is proposed, the investor can choose to either invest in it or withdraw her money from the SPAC. In either case, the game ends, so the opportunity to propose is unique. If no project is proposed by a deadline, the game also ends, and the investor gets her money back. The tension between the two players rests on two points. First, the sponsor has informational advantage over the investor. He always observes the type of a project, which is either good or bad, but the investor only with a probability. Second, their interests are only partially aligned. The investor, who bears the cost of investment, prefers a good project to no project and further to a bad project. The sponsor, who only enjoys the payoff of investment, prefers a good project to a bad project and further to no project.

We derive a unique sequential equilibrium of the SPAC game. Generically, the equilibrium consists of two stages: in the first stage, the sponsor proposes only good projects, and the investor always invests in the proposed project; in the second stage, the sponsor proposes all the good projects and a fraction of the bad ones he receives, and the investor invests contingent on the information she observes. Since the sponsor has only one chance to propose projects, the opportunity cost of proposing a project is his continuation value, which is the expected payoff of proposing the projects he receives in the future. Note that the sponsor can obtain a higher expected payoff from proposing a good project than proposing a bad one because the investment in a good project brings more to the sponsor and is also more likely to be approved by the investor. As a result, the sponsor with a good project must propose because at best he can receive another good project in the future. As for the sponsor with a bad project, waiting is double-edged: he may be better off if a good project arrives and may be worse off if no project arrives. As the SPAC approaches its deadline, the downside becomes more and more dominant, and thus the sponsor's continuation value decreases. At a point, the sponsor starts to find proposing a bad project desirable. Concerned about the poor quality of the proposed project on average, the investor spontaneously chooses to invest more conservatively based on her information over time. Such conservatism effectively reduces the sponsor's expected payoff of proposing a bad project and in turn helps discipline the sponsor. By and large, the equilibrium is consistent with the conventional wisdom that the incentive misalignment gives rise to a moral hazard problem of the sponsor and it intensifies as the SPAC gets closer to the deadline.

Based on the equilibrium, we then analyze the nature of the sponsor's moral hazard problem—the central friction in the game. The sponsor's moral hazard is curbed by two forces. The first is the investor's screening based on her noisy information, and the second is the sponsor's continuation value. More importantly, the two forces intertwined with each other. On one hand, due to their substitution relationship in equilibrium, the investor's screening is decreasing in the sponsor's continuation value. On the other hand, the investor' screening reduces the possibility of investment and thus stifles the accumulation of the sponsor's continuation value. As a result, the sponsor's continuation value follows a kind of self-reinforcing dynamics: its accumulation rate is positively correlated with its current level. An important lesson is that both sides of the partial alignment in players' incentives are crucial to the equilibrium dynamics: while the misalignment side induces moral hazard, the alignment side helps mitigate it.

Next, we explore the welfare implications of current SPAC practice. A popular opinion is that the investor benefits from her control right over investment because it not only allows her to avoid investment in some bad projects but also discourages the sponsor from proposing bad projects in the first place. However, we find that when the investor's information is sufficiently noisy, this control arrangement actually exacerbates the sponsor's moral hazard problem and reduces the investor's welfare. What the popular opinion misses is that the investor's screening makes waiting less attractive for the sponsor, which incentivizes him to propose more bad projects. When the sponsor has the control right, he proposes any project he receives near the deadline. Though hurting the investor, such undisciplined behavior results in rapid accumulation of the sponsor's continuation value. Hence, in a long period following the beginning of the game, the continuation value is high enough to prevent the sponsor from proposing any bad project, and the investor can fully enjoy the payoff of the good project that arrives. When the investor has the control right, the sponsor's continuation value accumulates more slowly in a self-reinforcing manner as pointed out above. In the case that the investor's information is very noisy, a low level of the sponsor's continuation value translates into a low accumulation rate in the absolute sense. Since the continuation value accumulates from 0, it will be trapped by the self-reinforcing dynamics at a low level for a long period. In one extreme, during the whole game, the investor will exert stringent screening, miss most good projects that arrive, and earn little profit in expectation. This analysis uncovers that the investor's equilibrium screening, which is optimal ex post after the sponsor proposes a project, is too stringent ex ante because it has a negative externality on the sponsor's continuation value. Hence, regarding the design of SPAC, a natural question is whether there is a way to rein in the screening to strike a balance between the two disciplining forces. We find that it is sometimes helpful to make the control right contingent on certain public assessment, e.g. credit rating or auditing trusted by both players. Specifically, the investor should own the control right only if the result of public assessment is below a threshold.

Another issue regarding current SPAC practice we explore is one proposal vs. multiple proposals. Motivated by the recent trend of SPAC, we model the investor's decision-making process as a tender offer, which restricts the sponsor to proposing at most one project in the game. An alternative is to allow the sponsor to continue searching and proposing after a proposal is rejected by the investor until the deadline. Notably, multiple proposals can be naturally implemented if the investor's decision making is structured as voting. On one hand, the coercive termination feature of one proposal eliminates potential investment opportunities and hurts both parties. On the other hand, it enables the investor's screening to curb the sponsor's moral hazard and result in less stringent screening in equilibrium, so both parties benefit from it. We find that the sponsor's welfare is always higher under one proposal but the investor's is ambiguous. This intuition justifies the recent transition from voting to tender offers from an equilibrium perspective.

Finally, we discuss several extensions of the model. First, we explicitly incorporate entrepreneurs into the model and consider their strategic behavior. Entrepreneurs can raise funds through either the SPAC or a standard IPO. The opportunity cost of tapping the SPAC is that the deal may not be approved by the investor and the IPO process is also delayed. Hence, the investor's screening effectively discourages entrepreneurs from tapping the SPAC and diminishes the flow of projects received by the sponsor. Second, we consider the sponsor's endogenous effort to search for projects. We find that as the SPAC approaches its deadline, the sponsor's equilibrium effort first increases due to declining continuation value and then decreases due to intensifying screening of the investor. The two extensions further stoke our concern that the investor's control right exacerbates the moral hazard problem and may backfire. Third, we consider the case of long-lived projects where the sponsor can possibly keep a project for future proposals. It turns out that such possibility does not alter the equilibrium dynamics in the baseline setup. Fourth, we extend the model to multiple investors. Now, the investment in a project requires the approval of sufficient investors. An investor can infer other investors' information through the threshold in equilibrium. The equilibrium is similar to that with only one investor but has richer dynamics.

The paper proceeds as follows. The remainder of this section reviews the related literature. Section 2 describes the baseline setup. Section 3 characterizes the equilibrium. Section 4 analyzes current SPAC practice and discuss the design of SPAC. Section 5 extends the baseline setup along several dimensions. Section 6 concludes the paper. All proofs are given in Appendix.

Related Literature. This paper mainly contributes to two strands of the literature. First, there is a growing empirical literature examining the development, trend and performance of SPACs. Gahng et al. (2021) examine SPAC performance and show that SPAC IPO investors earn positive 9.3% per year, while post-merger returns are significantly negative. They also document that there is no cost advantage of SPACs compared with traditional IPO. Dimitrova (2017) shows that SPAC performance is worse for deals announced near the two-year deadline, which is consistent with our theoretical prediction. Examining the factors that influence approval probability, Cumming et al. (2014) find that the presence of active investors in a SPAC is negatively correlated with approval probability . Klausner et al. (2020) show that the post-merger performance is negatively correlated with dilution and cash shortfall. Blomkvist and Vulanovic (2020) show that the SPAC volume and SPAC share of total IPOs are negatively correlated with VIX and time-varying risk aversion, implying market condition is a key factor in SPAC development.

There is little work on the theoretical side despite SPAC has become more and more important in recent years. Bai et al. (2021) provide a model with endogenous segmented markets, and argues that SPAC is welfare improving as it works as certification intermediaries for risky firms who were unserved by the traditional IPO. Chatterjee et al. (2016) consider a security design problem and argue warrants in SPACs can help to mitigate the moral hazard problem in project selection. Our focus is how the partial alignment in incentive between SPAC sponsors and investors shapes their interaction in a dynamic setup and its welfare implications in SPAC lifecycle. To the best of our knowledge, this is the first theoretical paper examining the SPAC lifecycle in a dynamic environment. We also contribute to the literature by providing discussions on counterfactuals. SPACs is relatively new and has received less attention compared to traditional IPO. With SPAC developing in a fast-changing environment, it's crucial to understand the current practice of SPACs as well as counterfactuals. Our discussion on control rights and one proposal vs multiple proposals sheds light on the design of SPACs.

Second, our paper contributes to the literature on delegation and authority in organizations (Crawford and Sobeli 1982; Aghion and Tirole 1997; Dessein 2002; Grenadier et al. 2016; Guo 2016). In this literature, the principal (the SPAC investor in our model) cannot commit to a decision rule and the allocation of control matters. There are several trade-offs identified in this literature, including the trade-off between informativeness vs bias (Dessein 2002) and information acquisition of different players (Aghion and Tirole 1997). In our discussion on control rights, we extend the model to the case when the sponsor has the control right and compare it with our baseline

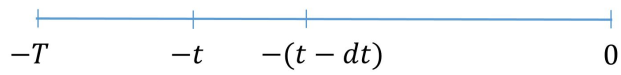

Figure 1: Time Flow

case when the investor makes the final decision. The allocation of control endogenously changes the shape of sponsor's bias in the SPAC lifecycle. When delegating the investment decision to the sponsor, the investment decision is efficient for a longer period but it deteriorates when the SPAC approaches the deadline. This new trade-off is a direct result of the dynamic nature and the hard deadline of decision making in our model, which is novel in the literature. As for dynamic setups, Grenadier et al. (2016) considers a model in which the principal exercises an option and relies on an informed but biased agent. Guo (2016) considers a dynamic delegation model with experimentation in which the principal and agent have different preferences on project riskiness. Guo (2016) is not a stopping-time game and thus fundamentally different from ours.

2 A Dynamic Model of SPAC

2.1 Model Setup

Consider a SPAC with one penniless sponsor (he) and one investor (she). They are both risk neutral and have common discount rate $r = 0$[3]. Motivated by the practice in reality, we model the SPAC as a finite-horizon continuous-time dynamic game unfolding over the period $[-T, 0]$. Figure 1 is a representation of the time flow. Both t and T are non-negative, and physical time moves forward as t decreases from T to 0. As we will show later, it's easier to consider our model backward, which corresponds to t increasing from 0 to T.

Projects Since the paper is primarily focused on the strategic interaction between the sponsor and the investor, we abstract away entrepreneurs' strategic behavior and assume an exogenous process of projects[4]. Per unit of time, the sponsor receives projects at the rate λ. The type of a project ω can be either good (G) or bad (B), and the probability (odds) of receiving a good project

[3] We assume no discounting merely to simplify the exposition. The main results hold for a positive discount rate.

[4] In Section 5.1, we explicitly model entrepreneurs' strategic behavior and examine its impact on the equilibrium dynamics.

is $\text{Prob}(G) = p_0$ ($\theta_0 = p_0/(1-p_0)$). The arrivals and the types of projects are independent over time. Both the good and bad projects require the same investment $I = 1$ and generate gross return R_G and R_B respectively. We made the following assumptions on R_G and R_B:

Assumption 1. *(1) (positive gross returns)* $R_G > 1 > R_B > 0$*; (2) (negative NPV)* $p_0 R_G + (1 - p_0) R_B < 1$.

The first assumption states that the investment in a good project generates a higher return than that in a bad project and both returns are positive. The second one resonates with the concern that potential SPAC targets are of poor quality on average.

When a project arrives, the sponsor decides whether to propose it to the investor. For the baseline setup, we assume that projects are short-lived. That is, if the sponsor does not propose the project he receives, the project will disappear or become unavailable immediately. With this assumption, the state of the sponsor with respect to whether he has a project and what type of project he has is completely independent over time[5].

The investor's decision making A salient feature of SPACs is that it is the investors who finally decide whether to make an investment. Traditionally, after the sponsor proposes a project, the investors vote on acquisition approval. The acquisition is approved if and only if a sufficient fraction of investors vote for it. However, in the recent wave of SPACs, tender offer becomes the most popular way to structure the investors' decision making. Shachmurove and Vulanovic (2017) claims that "these post financial crisis SPACs are almost exclusively structured as tender offers". Motivated by the trend, we model the sponsor's proposal as a tender offer. The investor can choose to either invest $I = 1$ and receive a pre-specified fraction of shares of the project, or withdraw from the SPAC. Because of the nature of a tender offer, the game ends immediately after a proposal. Essentially, the sponsor has only one opportunity to propose a project to the investor in the lifecycle of a SPAC.

Information Both the arrivals and the types of projects are observable to the sponsor but not to the investor[6] . When the sponsor proposes a project, the investor observes the true type of

[5] In Section 5.3, we study the case that projects are long-lived and thus the state of the sponsor is positively correlated over time.

[6] The assumption that arrivals are privately observed by the sponsor is not important. The equilibrium will be the same even if the arrivals are publicly observable.

	H	M	L
$\text{Prob}(\cdot; G)$	q	$1-q$	0
$\text{Prob}(\cdot; B)$	0	$1-q$	q

Table 1: The investor's observations

the project with the probability q and nothing otherwise. We denote the investor's observations as $\{H, M, L\}$, whose probabilistic structure follows Table 1. Hence, q stands for the quality of the investor's information, and $q < 1$ captures information asymmetry between the sponsor and the investor. Notice that with Assumption 1, if the sponsor proposes any project he receives, the investor will have negative expected profit of investing upon observing M.

The payoff structure Depending on the investment result, the sponsor's and the investor's payoffs follow Table 2. If the investor chooses to invest and the type of the project is ω, she receives u_ω and the sponsor receives $v_\omega = R_\omega - u_\omega$ from the project. If the investor chooses to withdraw from the SPAC, she keeps her money $I = 1$ and the sponsor receives 0. If the sponsor does not make a proposal by the time 0, the investor automatically withdraws. We make the following assumptions regarding the payoff structure:

Assumption 2. *(partial alignment)* $v_G > v_B > 0$, $u_G > 1 > u_B$.

This assumption stems from the contractual arrangement of SPAC: the shares granted to the sponsor is not contingent on the value of the project. Typically, the sponsor can obtain 20% of the shares of the target firm owned by the SPAC, and the investors the rest 80%. As a result, when comparing investing in a bad project with investing in no project, the sponsor prefers the former to the latter, while the investor opposite. As recognized by both the academia and practitioners, this preference misalignment underlies the fundamental moral hazard problem in SPACs[7]. On the other hand, it should not be ignored that the contractual arrangement also has an alignment side: both the sponsor and the investor prefer investing in a good project to investing in a bad project or no project. As shown later, both sides of the partial alignment play important roles in equilibrium dynamics.

[7] Aware that potential agency problems may discourage investors, some SPAC sponsors try to tie the shares they get more closely to the ex post value of the firm through deferred grant or clawback. Also, it becomes more popular to let the sponsor have some skin-in-the game. However, these remedies are still far from eliminating the misalignment.

Investment	G	B	withdrawal
The sponsor's payoff	v_G	v_B	0
The investor's payoff	u_G	u_B	1

Table 2: The payoff structure

Timeline Although the game is in continuous time, heuristically, conditional on the game continues at time $-t$, each instantaneous "period" $[-t, -(t-dt))$ consists of events occurring in the following order:

1. With the probability λdt, the sponsor receives a project and observes its type;
2. Receiving a project, the sponsor can propose it or not;
3. If the sponsor proposes a project, the investor receives a signal and chooses to invest in the project or withdraw; then the game ends, and both players receive their payoffs;
4. If the sponsor does not propose a project, the game continues to $-(t-dt)$.

2.2 Equilibrium Concept

We focus on the sequential equilibria of the game. First, we characterize the players' strategies and beliefs. Since the game has a finite horizon, time is naturally a state variable that their strategies are based on. The sponsor has only one action in the game: whether to propose the project he receives. Hence, his strategy can be characterized by $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$, where $\alpha_\omega(-t)$ represents the probability that the sponsor proposes the project of the type ω at the time $-t$. The investor also has only one action in the game: whether to invest in the project proposed by the sponsor based on her signal. Therefore, her strategy can be characterized by $(\eta_s(-t))_{s\in\{H,L,M\}}$, where $\eta_s(-t)$ represents the probability that the investor invests at the time $-t$ when observing the signal s.

The players' beliefs can be characterized accordingly. Let $(\tilde{\eta}_s(-t))_{s\in\{H,L,M\}}$ be the sponsor's belief about the investor's strategy. Then by proposing a project of the type ω to the investor at $-t$, the sponsor's expected payoff is

$$F_\omega(-t) \equiv \begin{cases} [q\tilde{\eta}_H(-t) + (1-q)\tilde{\eta}_M(-t)]\, v_G, & \text{if } \omega = G \\ [(1-q)\tilde{\eta}_M(-t) + q\tilde{\eta}_L(-t)]\, v_B, & \text{if } \omega = B \end{cases}.$$

Let $\tilde{\theta}(-t)$ be the investors' prior belief of the odds of a good project before observing the signal. As required by sequential equilibria, these beliefs should be consistent with the strategies on the equilibrium path according to Bayes' rule. However, in this model, sequential equilibria have no effective restriction on the beliefs off the equilibrium paths. Specifically, if $\alpha_G(-t) = \alpha_B(-t) = 0$ at a time $-t$, $\tilde{\theta}(-t)$ can take any nonnegative values. This gives rise to multiplicity of equilibria[8]. To obtain sharper predictions of the equilibrium, we impose D1 refinement: the investor believes that the project must be good if it is proposed by the sponsor at a time when no project should be proposed in equilibrium.

Below is the equilibrium concept used throughout the paper.

Definition 1. An (sequential) equilibrium consists of the sponsor's proposal strategy $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$, the investor's investment strategy $(\eta_s(-t))_{s\in\{H,L,M\}}$, the sponsor's belief $(\tilde{\eta}_s(-t))_{s\in\{H,L,M\}}$, and investor's belief $\tilde{\theta}(-t)$ such that at any time $-t \in [-T, 0]$ and conditional on no proposal before $-t$, the following conditions hold:

1. $(\alpha_\omega(-\tau))_{\omega\in\{G,B\}}$ after $-t$ maximizes the sponsor's continuation value at $-t$:

$$V(-t) = \max_{(\alpha_\omega(-\tau))_{\omega\in\{G,B\}}} \int_0^t P(-\tau; -t) \cdot \lambda \left[p_0 \alpha_G(-\tau) \cdot F_G(-\tau) + (1-p_0)\alpha_B(-\tau) \cdot F_B(-\tau)\right] d\tau,$$

   where $P(-\tau; -t) \equiv e^{-\int_\tau^t \lambda[p_0\alpha_G(-\xi)+(1-p_0)\alpha_B(-\xi)]d\xi}$ is the probability that the game still continues at time $-\tau > -t$ conditional on that the game continues at time $-t$.

2. For any $s \in \{H, M, L\}$, the investor's investment strategy $\eta_s(-t)$ maximizes her expected profit based on the prior belief $\tilde{\theta}(-t)$ and the signal s:

$$\eta_s(-t)\left\{\frac{\tilde{\theta}(-t)\frac{\text{Prob}(s;G)}{\text{Prob}(s;B)}}{1+\tilde{\theta}(-t)\frac{\text{Prob}(s;G)}{\text{Prob}(s;B)}}(u_G - u_B) + u_B - 1\right\}.$$

3. Rational beliefs and D1 refinement:

   (a) $\tilde{\eta}_s(-t) = \eta_s(-t)$ for all $-t$ and $s \in \{H, L, M\}$;

   (b) $\tilde{\theta}(-t) = \frac{p_0}{1-p_0}\frac{\alpha_G(-t)}{\alpha_B(-t)}$ for all $-t$ satisfying $\alpha_G(-t) + \alpha_B(-t) > 0$;

   (c) $\tilde{\theta}(-t) = +\infty$ if $\alpha_G(-t) = \alpha_B(-t) = 0$.

[8] Besides the equilibrium we derive later, another obvious equilibrium is that $\alpha_G(-t) = \alpha_B(-t) = 0$ and $\tilde{\theta}(-t) = 0$ for all time points.

3 Model Solution

3.1 Equilibrium Characterization

We first analyze the investor's problem. When the investor observes the signal H (L), her posterior probability of the project proposed being good becomes 1 (0), and her net payoff from investing in the project is $u_G - 1 > 0$ ($u_B - 1 < 0$). Thus her equilibrium strategy must be $\eta_H(-t) = 1$ and $\eta_L(-t) = 0$ for all $-t$. To characterize the investor's equilibrium strategy, we can focus on that when she observes the signal M, i.e., $\eta_M(-t)$. For simplicity, we get rid of the subscript of η_M, and let $\eta(-t) \equiv \eta_M(-t)$. It is easy to see that the investor's problem can be reduced to

$$\max_{\eta(-t)} \eta(-t) \left\{ \tilde{\theta}(-t) - \frac{1 - u_B}{u_G - 1} \right\},$$

where $\tilde{\theta}(-t)$ is the investor's posterior belief of the odds of a good project. Then we obtain the following lemma.

Lemma 1. *In equilibrium, at any time $-t$,*

1. $\eta_H(-t) = 1$, *and* $\eta_L(-t) = 0$ *;*
2. *When* $\tilde{\theta}(-t) > (<) \frac{1-u_B}{u_G-1}$, $\eta(-t) = 1\,(0)$*; when* $\tilde{\theta}(-t) = \frac{1-u_B}{u_G-1}$, $\eta(-t) \in [0,1]$.

Next, we turn to the sponsor's problem. According to Lemma 1 and rational beliefs in equilibrium, if the sponsor proposes a project of type ω at time $-t$, his expected payoff is

$$F_\omega(-t) = \begin{cases} [q + (1-q)\,\eta(-t)]\, v_G, & \text{if } \omega = G \\ (1-q)\,\eta(-t)\, v_B, & \text{if } \omega = B \end{cases}.$$

At any time $-t$, the sponsor's continuation value $V(-t)$ satisfies the HJB equation

$$\frac{dV(-t)}{dt} = \max_{\alpha_G(-t),\alpha_B(-t)} \lambda p_0 \cdot \alpha_G(-t) \cdot [F_G(-t) - V(-t)] + \lambda (1-p_0) \cdot \alpha_B(-t) \cdot [F_B(-t) - V(-t)]. \tag{1}$$

In addition, at the last instant of the game, it is almost sure that the sponsor will not receive a project, so the continuation value at $-t = 0$ must be 0, i.e., $V(0) = 0$. eq. (1) reflects an important feature of the game: the sponsor has only one opportunity to propose a project. When proposing a project of the type ω at $-t$, the sponsor can get expected payoff $F_\omega(-t)$. However, he also loses the opportunity to receive and propose new projects in the future, whose value amounts to $V(-t)$ in expectation. Therefore, the sponsor's equilibrium strategy $\alpha_\omega(-t)$ must satisfy

$$\alpha_\omega(-t)\begin{cases} = 1 & \text{if } F_\omega(-t) - V(-t) > 0 \\ \in [0,1] & \text{if } F_\omega(-t) - V(-t) = 0 \\ = 0 & \text{if } F_\omega(-t) - V(-t) < 0 \end{cases}$$

for $\omega \in \{G, B\}$.

A critical observation of the game is that the sponsor always has more incentive to propose a good project than a bad one. On one hand, $F_G(-t) > F_B(-t)$ always holds because a good project not only gives the sponsor a higher payoff than a bad one but also is more likely to be approved by the investor. On the other hand, the opportunity cost of proposing a project at time $-t$ is $V(-t)$, which is independent of the type of the project that the sponsor receives. An implication of the observation is that in equilibrium, it is always strictly better for the sponsor to propose a good project than not. The sponsor's continuation value is decreasing over time because as the time passes, he is less likely to receive and propose a good project.

Lemma 2. *In equilibrium, for any $-t$, $F_G(-t) > V(-t)$. Further, $\alpha_G(-t) = 1$, and $V(-t)$ strictly decreases to 0 as $-t$ increases to 0.*

Since the sponsor always propose the good project he receives, the perceived quality of the proposed project depends on his incentive of proposing bad projects. Lemma 3 implies that whenever the sponsor receives a bad project, he must choose not to propose it with positive probability. This relies on the key assumption that the potential projects of SPACs have negative NPV on average, i.e., $p_0R_G + (1-p_0)R_B < 1$. If the sponsor surely proposes the bad project he receives at a time point, the investor must withdraw surely when observing M because she has negative expected profit of investing. Then the sponsor should have no incentive to propose a bad project. Hence, this situation cannot take place in equilibrium.

Lemma 3. *When $V(-t) < (1-q)v_B$ and $-t < 0$, $\alpha_B(-t) \in (0,1)$. When $V(-t) > (1-q)v_B$, $\alpha_B(-t) = 0$.*

Combining Lemma 1, Lemma 2, and Lemma 3, we obtain a unique equilibrium of the game.

Proposition 1. *The unique equilibrium of the SPAC game has potentially two stages, the transition time between which is $-t^*$.*

- *The second stage spans the period* $(-t^*, 0]$*, in which*
  - *the investor's equilibrium strategy* $\eta(-t)$ *makes the sponsor indifferent to whether to propose a bad project or not, i.e.,*

$$V(-t) = F_B(-t) = (1-q)\eta(-t)v_B;$$

  - *the sponsor's equilibrium strategy* $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$ *satisfies* $\alpha_G(-t) = 1$ *and makes the investor indifferent to whether to invest or withdraw when observing M, i.e.,*

$$\frac{p_0}{1-p_0}\frac{\alpha_G(-t)}{\alpha_B(-t)} = \frac{1-u_B}{u_G-1};$$

  - *the sponsor's continuation value satisfies*

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)]$$

  with the boundary condition $V(0) = 0$*.*
- *The first stage spans the period* $[-T, -t^*)$*, in which*
  - *the investor always invest when observing M, i.e.* $\eta(-t) = 1$*;*
  - *the sponsor proposes only good projects, i.e.,* $\alpha_G(-t) = 1$ *and* $\alpha_B(-t) = 0$*;*
  - *the sponsor's continuation value satisfies*

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [v_G - V(-t)].$$

- *The transition time* $-t^*$ *satisfies* $V(-t^*) = (1-q)v_B$*, and* $V(-t)$ *is continuous at* $-t^*$*. If* $-T \geq -t^*$*, the first stage will be degenerate, and the equilibrium has only the second stage.*

The misalignment of the two players' payoffs is key to the equilibrium dynamics. Due to the finite horizon of SPAC, as time passes, the sponsor has less chance to receive a project, and thus his continuation value decreases. Note that the continuation value is also the opportunity cost of proposing a project, which dampens the sponsor's desire to propose a bad project. In the early stage of the game, the continuation value is high enough to prevent the sponsor from proposing any bad project, even though the investor imposes the least stringent screening, $\eta(-t) = 1$. Later on, the sponsor starts to find proposing a bad project desirable. Because of the poor quality of potential projects on average, the investor is concerned about an undisciplined sponsor and spontaneously chooses to invest more conservatively. Such conservatism imposes more stringent screening, $\eta(-t) < 1$, which in turn helps discipline the sponsor.

However, the alignment of their payoffs also plays an important role here. Although the investor imposes more stringent screening in the future, the sponsor's continuation value, which accumulates from potential investment in the future, can always be no less than his expected payoff of proposing a bad project currently. That is because the sponsor may receive good projects in the future. According to the payoff structure of SPACs, a good project gives the sponsor a higher payoff than a bad one. Moreover, since the investor also prefers a good project to a bad one, her screening automatically makes investment in the former more likely than that in the latter. This further enhances the attractiveness of waiting for a good project relative to proposing a bad project currently.

3.2 The Welfare

Proposition 2 provides a characterization of the sponsor's welfare and the investor's welfare based on the properties of the equilibrium. Denote the investor's continuation value at $-t$ by $U(-t)$. Then the sponsor's welfare and the investor's welfare are $V(-T)$ and $U(-T)$ respectively.

Proposition 2.

- *Given the investor's equilibrium strategy $\eta(-t)$, $V(-T)$ is equal to the sponsor's expected payoff if he proposes only good projects to the investor, i.e.,*

$$V(-T) = v_G \int_{t=0}^{T} \lambda p_0 e^{-\lambda p_0 (T-t)} \left[q + (1-q)\eta(-t)\right] dt. \tag{2}$$

- *Given the sponsor's equilibrium strategy $\alpha_G(-t) = 1$ and $\alpha_B(-t)$, $U(-T)$ is linear in the unconditional probabilities that the sponsor proposes good projects in the two stages, i.e.,*

$$U(-T) = (u_G - 1) \cdot (P_1^* + q \cdot P_2^*) + 1, \tag{3}$$

where

$$P_1^* = \int_{\min\{T,t^*\}}^{T} \lambda p_0 e^{-\lambda p_0 (T-t)} dt,$$

$$P_2^* = \int_{t=0}^{\min\{T,t^*\}} \lambda p_0 e^{-\lambda p_0 (T-t)} \cdot e^{-\lambda (1-p_0) \int_t^{\min\{T,t\}} \alpha_B(-\tau) d\tau} dt.$$

Since the sponsor is always indifferent to whether to propose a bad project or not in the second stage, we can use the equilibrium path in which the sponsor never proposes a bad project to calculate his welfare. A useful property of the sponsor manifested by the representation (2) is that his welfare depends on only how likely a proposed good project is invested by the investor. The probability is $q+(1-q)\eta(-t)$ at $-t$, which depends on the quality of the investor's information as well as her screening in equilibrium.

The statement about the investor's welfare stems from the observation that the investor always breaks even in expectation except that she knows the proposed project is surely good. In the first stage, only good projects are proposed, and the probability that it happens is P_1^*. In the second stage, the investor's expected profit is equal to her outside option 1 when observing M or L, and she knows the project is surely good when observing H. With probability P_2^*, the sponsor proposes a good project in the second stage, and conditional on that, the investor observes H with the probability q. Note that we always have $V(-t^*) = (1-q)v_B$ at the start of second stage. The representation (3) implies that the key elements in investor's welfare are: the length of the second stage t^*, the probability that the sponsor proposes a bad project in the second stage $\alpha_B(\cdot)$, and the quality of her information q.

3.3 Moral Hazard in Equilibrium

In this subsection, we take a closer look at the sponsor's moral hazard problem. Notice that the sponsor acts in the investor's best interest in the first stage and his moral hazard problem is present in only the second stage. Hence, there are two dimensions regarding the degree of his moral hazard in equilibrium: the duration of moral hazard, which is represented by the length of the second stage t^*, and the intensity of moral hazard, which is represented by the probability of proposing a bad project in the second stage $\alpha_B(-t)$.

Proposition 3.

- *t^* satisfies*

$$\left[e^{\lambda p_0\left(\frac{v_G}{v_B}-1\right)t^*}-1\right]\frac{1}{\frac{v_G}{v_B}-1}\cdot q\cdot v_G=(1-q)\cdot v_B. \tag{4}$$

  t^* is decreasing in v_G/v_B and q.

- *For $t<t^*$,*

$$\alpha_B(-t)=\frac{p_0}{1-p_0}\frac{u_G-1}{1-u_B}. \tag{5}$$

The length of the second stage depends on two factors. The first is the sponsor's continuation value, and the second is the maximum expected payoff the sponsor can receive from proposing a bad project. He can receive at most $(1-q)v_B$ from proposing a bad project, since the investor must withdraw when observing the signal L. In the first stage, the former is greater than the latter but keeps decreasing. When the former meets the latter, the second stage starts, and lasts until time $-t=0$. When we look at the model backward from time $-t=0$, the sponsor's continuation value is simply all the value $dV(-t)/dt$ accumulated from time 0 to time $-t$. As implied by Proposition 2, the sponsor's continuation value essentially relies on only the proposal of good projects and consists of two parts. First, upon observing M, the investor invests in the project with the probability $\eta(-t)$ at $-t$, which results in investment in good projects occurring at the rate of $\lambda p_0(1-q)\eta(-t)$. This part corresponds to

$$\left[e^{\lambda p_0\left(\frac{v_G}{v_B}-1\right)t^*}-1\right]\frac{1}{\frac{v_G}{v_B}-1}$$

in eq. (4). Second, upon observing H, the investor invests in the project with the probability 1, which results in investment in good projects occurring at the rate of $\lambda p_0 q$. This part corresponds to the q in the left-hand side of eq. (4).

Rather than an additive relationship implied by their origination, the parts are convoluted in an multiplicative manner. Note that in equilibrium, $\eta(-t)$ makes the sponsor indifferent to whether to propose a bad project or not, so it satisfies

$$(1-q)\eta(-t)\cdot v_B = V(-t).$$

That means, the first part accumulates by an amount proportional to the level of the sponsor's continuation value. Due to such self-reinforcing dynamics, the sponsor's continuation value becomes very sensitive to q.

The probability of proposing a bad project $\alpha_B(-t)$ is actually a constant in the second stage. The investor is indifferent between withdrawing and investing upon observing signal M, so her posterior belief upon observing signal M must be a constant. As a result, $\alpha_B(-t)$ depends on the quality of the project pool and the investor's payoff structure.

4 Welfare Implications and the Design of SPAC

4.1 The control arrangement

As shown above, a typical SPAC suffers from moral hazard problems since the sponsor and the investor are not fully aligned about what projects should be invested in. Such systematic misalignment is actually rare in other delegated investment vehicles usually considered to be comparable to SPAC. Regarding private equity, hedge fund, and mutual fund, the sponsor's objective is primarily to maximize the value of the whole fund and thus consistent with the investor's. This is also an important reason why we observe that only SPAC investors can directly decide whether to invest in a project. Such arrangement of investment control rights is meant to mitigate moral hazard problems and facilitate investors' participation in the game in the first place.

In this subsection, we present a welfare analysis of SPAC with respect to investment control rights. We characterize the equilibrium when the sponsor can directly decide whether to invest. Surprisingly, our analysis implies that in some cases, the SPAC investor can be better off if the sponsor has the control right.

Suppose that the sponsor can directly decides whether to invest. Since the sponsor's proposal guarantees investment, his payoff is v_ω if he proposes a project of the type ω. Let $V_s(-t)$ represent his continuation value at time $-t$. It is easy to see that as the time passes, the continuation value must be weakly decreasing and always smaller than v_G. At the last instant of the SPAC life cycle, the continuation value must be 0. Similar to the case that the investor has the control right, the game is divided into two stages in equilibrium. Denote the transition time as $-t_s^*$. In the first stage where $-t < -t_s^*$, $v_B < V_s(-t) < v_G$, so the sponsor proposes only the good project he receives. In the second stage where $-t > -t_s^*$, $V_s(-t) < v_B < v_G$, and the sponsor proposes any project he receives. Let $U_s(-t)$ represent the investor's continuation value at the time $-t$. We readily obtain the following properties about the equilibrium.

Lemma 4.

- *t_s^* is finite and satisfies*

$$\left(1 - e^{-\lambda t_s^*}\right)[p_0 v_G + (1 - p_0) v_B] = v_B$$

- *There exists $T_s^* > 0$ such that $U_s(-T) > 1$ if and only if $T > T_s^*$.*

The first point of Lemma 4 states that although the sponsor has full discretion over investment, he only acts at odds with the investor's interest in a later period of the game. Similar to that in the baseline setup, the equilibrium has two stages. In the second stage, the sponsor's continuation value accumulates at the rate of $\lambda\left[p_0 v_G + (1-p_0) v_B\right]$ while his expected payoff from proposing a bad project is always v_B. The second stage starts when the former meets the latter, and lasts until $-t = 0$. The key force behind the equilibrium is that the sponsor also prefers a good project to a bad one as the investor does. With only one opportunity to invest, if he expects that the remaining time allows him to receive a good project with a sufficiently high probability, he would prefer to forgo the bad project at hand despite the risk that he may end up with no project.

The second point stems from the following fact: in the second stage, the sponsor invests in all the projects he receives, so the investor loses money from investment in expectation because of the poor quality of potential SPAC projects on average, i.e.,

$$p_0 u_G + (1-p_0) u_B < 1.$$

In the first stage, the sponsor invests in only the good projects he receives, so the investor gets positive profit from investment in this stage. As a result, when T is large enough, going backward from the last instant $-t = 0$ to $-t = -T$, the investor's continuation value $U_s(-t)$ first decreases and then increases.

Next, we focus on how the ownership of the control right affects the two players' welfare.

Proposition 4.

- *$V_s(-T) > V(-T)$ for any T.*
- *Suppose $T > T_s^*$. There exists q_s^* such that $U_s(-T) > U(-T)$ if and only if $q < q_s^*$.*

The first point is straightforward. When the sponsor has full control over investment, his expected payoff must dominate his payoff in the baseline setup. The second point implies that if the investor's information is very noisy, the investor can be better off if the sponsor has the control right and T is large enough. The key to the result is that when q is small, the investor's control right exacerbates the sponsor's moral hazard problem and prolongs the second stage.

Next, we explain how the investor's control right affects the length of the second stage. When the investor has control right, at any time $-t$, a proposed project will be rejected with probability 1 if the signal L is observed and will be rejected with the probability $1 - \eta(-t)$ if the signal M is observed. Compared to the case when the sponsor has the control right, such potential rejection directly reduces the sponsor's expected payoff from proposing a bad project to $(1-q)\eta(-t)\cdot v_B$ and thus reduces its maximum to $(1-q)\cdot v_B$. Apparently, this direct effect shortens the second stage. It is consistent with the conventional wisdom that with the control right, the investor's profit-maximizing decision can naturally discipline the sponsor's behavior. However, the rejection also impedes the accumulation of the sponsor's continuation value. Recall that according to Proposition 2, the accumulation essentially comes from only the proposals of good projects. Hence, at any time $-t$ in the second stage, the sponsor's continuation value accumulates at the rate of $\lambda p_0 [q+(1-q)\eta(-t)]\cdot v_G$ in the case when the investor has the control right, as opposed to $\lambda [p_0 v_G + (1-p_0)v_B]$ in the case when sponsor has the control right. So when the investor has control right, the sponsor's continuation value accumulates more slowly, which can potentially make the second stage longer.

Then what is the net effect of the investor's control right when q is small? The reduction in the maximum expected payoff from proposing a bad project is proportional to q and thus small, but that in the accumulation of the sponsor's continuation value could be very large. As pointed out in Section 3.3, the sponsor's continuation value follows a self-reinforcing dynamics. With a small q, a low level of the sponsor's continuation value directly translates into a low accumulation rate. Since the accumulation starts at 0, the self-reinforcing dynamics essentially trap it at a low level for a long period. As a result, it takes long for the sponsor to accumulate sufficient continuation value to leave the second stage.

In this SPAC setup, there are potentially two forces that determine the degree of the sponsor's moral hazard problem in equilibrium. As argued previously, the investor and the sponsor are partially aligned: they both prefer good projects to bad projects. Such partial alignment, which works through the accumulation of the sponsor's continuation value, naturally motivates the sponsor to act in the investor's interest to some extent. When the investor has the control right, she exerts screening based on her information. The screening directly disciplines the sponsor's behavior, yet it also dampens the effect of partial alignment. Our analysis suggests that the investor's equilibrium screening, which is optimal ex post after the sponsor proposes a project, is too stringent ex ante because the investor does not consider the negative externality on the sponsor's continuation value. The precision of the investor's information determines which side of the excessive screening dominates.

4.2 Public assessment and contingent control right

Public assessment of assets or projects plays an important role in various financing activities. For example, credit rating in bond issuance and auditing in syndicated loans. Apart from providing trustworthy information for investors, another potential function of public assessment is to provide public signals used for contracting. As implied by our previous analysis, giving the control right to either party may both incur severe welfare loss. In this subsection, we show that making the control right contingent on public signals can be more favorable in some situations.

Suppose there is a public assessment agency that always publicly, truthfully discloses what it observes about the project proposed by the sponsor. Its information structure is similar to the investor's: if the project is good (bad), it observes H (L) with the probability $\hat{q}$ and M otherwise. We focus on monotone contingent allocation of the control right[9]: the sponsor can decide whether to invest if and only if the public signal is more favorable than a threshold. Notice that the two players always prefer the same decision upon observing H. If the sponsor (investor) has the control right upon observing M and L, the case is equivalent to that he (she) has the full control right. Therefore, we only need to deal with the case that the sponsor has the control right when the public signal is M and the investor has the control right when it is L.

It is straightforward to see that the sponsor's proposal is certainly rejected when the public signal is L. Hence, his expected payoff is v_G if he proposes a good project and $(1-\hat{q})v_B$ if he proposes a bad one. Let $V_c(-t)$ represent his continuation value at $-t$. The game is divided into two stages in equilibrium. Denote the transition time as $-t_c^*$. In the first stage where $-t < -t_c^*$, $(1-\hat{q})v_B < V_c(-t) < v_G$, so the sponsor proposes only the good projects he receives. In the second stage where $-t > -t_c^*$, $V_c(-t) < (1-\hat{q})v_B < v_G$, so the sponsor proposes any project he receives. Let $U_c(-t)$ represent the investor's continuation value at the time $-t$. We obtain the following results.

Proposition 5.

- *The sponsor's welfare is increasing in the control right he has, i.e.,* $V_s(-T) > V_c(-T) > V(-T)$.
- *The investor's welfare is higher when the control right is contingent than when the sponsor has the control right, i.e.,* $U_c(-T) > U_s(-T)$.

[9] It can be shown that other allocation is weakly dominated by this family of allocations.

- *The total welfare of the two players is higher when the control right is contingent than when the sponsor has the control right, i.e.,* $V_c(-T) + U_c(-T) > V_s(-T) + U_s(-T)$.

The first point is straightforward: more control right is always beneficial to the sponsor. The second point says the investor is always better off with the contingent control right than none. Compared to no control right, the contingent control right effectively rejects the investment in bad projects with the probability $\hat{q}$ (when the public signal is L). First, it increases the investor's expected payoff at each instant in the second stage. Second, it also shortens the length of the second stage. The sponsor's expected payoff from proposing a bad project is reduced by a factor of $\hat{q}$ to $(1-\hat{q})v_B$, and his accumulation of continuation value is also reduced to $\lambda\,[p_0 \cdot v_G + (1-p_0)\cdot(1-\hat{q})v_B]$. The reduction in the latter is disproportionately low relative to the former because the former depends on only the investment in bad projects while the latter depends on that in both types of projects. This property also holds for more general signal structures: the beneficial disciplining effect of allowing the investor to reject the investment when observing sufficiently unfavorable signals probably outweighs its adverse effect on the accumulation of the sponsor's continuation value. For the same reason, the total welfare of the two players is higher with the contingent control right because it reduces investment in bad projects without affecting that in good projects.

The comparison between the contingent control right and the investor having the control right is ambiguous. Besides the forces related to control rights discussed in Section 4.1, it also depends on the quality of the public information compared to the investor's ($\hat{q}$ vs. q).

4.3 One proposal vs. multiple proposals

Motivated by the recent trend of SPAC, we model the investor's decision-making process as tender offers for the baseline setup. Once the sponsor proposes a project, the game enters into the investor's decision making stage and ends immediately after that. Hence, the sponsor has only one opportunity to propose projects before the deadline. Another possible, also natural way to structure a SPAC is to let the sponsor continue to search for projects until the deadline if the investor is not willing to invest in the current one. Then the sponsor will essentially have multiple opportunities to propose. For convenience, we regard the two regimes as one proposal and multiple proposals respectively. In this subsection, we examine the impact of allowing multiple proposals on the equilibrium and the players' welfare.

Notably, multiple proposals can be naturally implemented if the investor's decision making is structured as voting, which was a very popular practice until recently. Voting in SPACs proceeds as follows. After the sponsor proposes a project, the investors vote on acquisition approval. If a sufficient fraction of investors vote for it, the deal is approved. Then the investors who vote against the deal are offered the right to redeem their shares[10]. Investors who are not offered or do not exercise the right will invest. If the deal is not approved, the SPAC will continue, and the sponsor searches for new projects. In our single-investor setup, the investor will either approve the deal and invest, or disapprove it and let the SPAC continue, consistent with the regime of multiple proposals. Therefore, the comparison between the two regimes also sheds light on that between tender offers and voting.

We consider a derivation of our baseline setup and assume that the sponsor can continue to search if the proposed project is rejected. All other assumptions are unchanged. Let $V_v(-t)$ represent the sponsor's continuation value at $-t$ in this new game. Likewise, $V_v(-t)$ is weakly decreasing, always smaller than v_G, and equal 0 at the deadline of the SPAC. By proposing a project of the type ω at $-t$, the sponsor enjoys v_ω if the investor approves the deal and $V_v(-t)$ otherwise. Hence, given the investor's strategy, his marginal benefit of proposing a project of the type ω is proportional to $v_\omega - V_v(-t)$. To obtain sharp equilibrium prediction, we assume that the sponsor does not use weakly dominated strategies. That is, he proposes a project of the type ω at $-t$ with the probability 1 if $v_\omega - V_v(-t) > 0$ and 0 if $v_\omega - V_v(-t) < 0$. It is easy to see the game is still divided into two stages in equilibrium. Denote the transition time as $-t_v^*$. In the first stage where $-t < -t_v^*$, $v_B < V_v(-t) < v_G$, so the sponsor proposes only the good project he receives. In the second stage where $-t > -t_v^*$, $V_v(-t) < v_B < v_G$, so the sponsor proposes any project he receives. Let $U_v(-t)$ be the investor's continuation value at $-t$. We obtain the following results.

Proposition 6. *The sponsor always has a lower welfare under multiple proposals than under one proposal, i.e.,* $V_v(-T) < V(-T)$. *But the comparison about the investor's welfare between the two regimes is ambiguous.*

Surprisingly, the sponsor is worse off under multiple proposals. Since the first stage proceeds in the same way under both regimes, to understand the intuition, we can focus on the second. On one hand, under multiple proposals, his continuation value accumulates at a lower rate in the second

[10] This is required by stock exchange listing rules. In many cases, SPACs offer all investors the redemption rights.

stage, which is

$$\begin{aligned}\frac{dV_v(-t)}{dt} &= \lambda p_0 \cdot q\,[v_G - V_v(-t)] \\ &= \lambda p_0 \cdot [qv_G + (1-q)\,V_v(-t) - V_v(-t)]\,,\end{aligned}$$

as opposed to

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [qv_G + (1-q)\,\eta\,(-t)\,v_G - V(-t)]$$

under one proposal. The difference between the two accumulation rates is that when M is observed, the sponsor receives $V(-t)$ in expectation under multiple proposals while he receives v_G with the probability $\eta(-t)$ under one proposal. Recall that $(1-q)\eta(-t) \cdot v_B = V(-t)$, so

$$\eta\,(-t)\,v_G = \frac{v_G}{(1-q) \cdot v_B} V(-t) > V(-t).$$

This simple observation relies on two points. First, since the signal L has helped screen out the fraction q of bad projects, the investor can exert lesser screening when observing M. Second, the sponsor strictly prefers a good project to a bad one. On the other hand, the sponsor needs a higher continuation value to leave the second stage under multiple proposals, $V_v(-t) = v_B$, as opposed to $V(-t) = (1-q)v_B$ under one proposal. Hence, allowing multiple proposals also prolongs the second stage.

The underlying economic intuition is that the coercive termination feature of one proposal enables the investor's screening to have ex ante disciplining effect on the sponsor. Under multiple proposals, the investor's disapproval when observing M or L cannot suppress the sponsor's incentive to propose bad projects at all because disapproval is not worse than not proposing. To protect herself from the sponsor's undisciplined behavior, the investor has to reject any investment unless she observes the clear-cut good signal H. As pointed out in Section 4.1, this rational response further restricts the accumulation of the sponsor's continuation value and prolongs the second stage. This intuition also helps justify the recent transition from voting to tender offers from an equilibrium perspective.

However, the comparison about the investor's welfare is ambiguous. Allowing multiple proposals affects the welfare in two opposite ways. On one hand, it prolongs the less efficient second stage. On the other hand, it increases the rate at which the investor's continuation value accumulates in the second stage. In both regimes, the investor can earn the positive profit $u_G - 1$ in the second stage only when she observes H. Under multiple proposals, the game ends only in this event by the deadline, while under one proposal, the game may end when she observes M or L. Hence, this event occurs more likely under multiple proposals.

5 Extensions

5.1 Strategic entrepreneurs

In the baseline setup, we deliberately abstract away from entrepreneurs' strategic behavior to better focus on the strategic interaction between the sponsor and the investor. Usually, the entrepreneurs can choose to bring the project public through either SPAC or standard IPO, and in more general settings, entrepreneurs can choose other financing strategies. In our baseline setup, the equilibrium features a (weakly) decreasing probability of approval for both good and bad projects. Then in a more general setting when entrepreneurs are strategic, the choice between SPAC and standard IPO will be endogenous, and thus the supply of projects from entrepreneurs may not be constant in SPAC lifecycle. In this subsection, we introduce strategic entrepreneurs into the model and examine its impact on the equilibrium dynamics.

There are many entrepreneurs, each of whom is endowed with one project. Projects can be either good (G) or bad (B), and the fraction of good projects is p_0. Each entrepreneur observes the type of her own project. There is only one SPAC in the market operating in the way modeled in the baseline setup. At each instantaneous "period" $[-t, -(t-dt))$, a liquidity shock arrives with probability λdt, and a randomly chosen entrepreneur needs to raise $I = 1$ to continue her project. If the project is not funded instantly, it may fail or shrink over time. The entrepreneur hit by a liquidity shock can choose to bring her project public through either SPAC or standard IPO. There are three possible scenarios:

1. she chooses IPO directly;
2. she taps SPAC, and the project is funded by the SPAC investor;
3. she taps SPAC, but the project is not funded by the SPAC investor and she turns to IPO.

Denote an entrepreneur's payoff as π_{IPO} if the project is funded through IPO directly and as π_{SPAC} if it is funded through SPAC. In the case that she chooses IPO after unsuccessful SPAC financing, her payoff is $\rho \cdot \pi_{IPO}$. We assume $\rho < 1$ because preparing for SPAC delays the project's IPO process and the project may fail or shrink due to lack of funding during that period. We do not impose a particular probabilistic structure on ρ, π_{SPAC}, and π_{IP0}: they can vary with projects in any reasonable way. The relationship between π_{IPO} and π_{SPAC} is ambiguous and depends heavily on project specifics. π_{SPAC} could be greater than π_{IPO} for several reasons. For example, projects can be funded through SPAC more quickly; projects that cannot access standard IPO may go public through SPAC.

Due to the stringent screening process and regulatory requirement of IPO, bad projects can hardly be funded through it, so we assume $\pi_{IPO} = 0$ for bad projects. Then it is easy to see that entrepreneurs with bad projects will always tap SPAC. Now consider those with good projects. Suppose an entrepreneur expects that if she taps SPAC, her project can be funded through SPAC with the probability x. Then she will tap SPAC if and only if

$$\begin{aligned} & x \cdot \pi_{SPAC} + (1-x) \cdot \rho \pi_{IPO} > \pi_{IPO} \\ \Leftrightarrow & \frac{(1-\rho)\pi_{IPO}}{\pi_{SPAC} - \rho \pi_{IPO}} < x. \end{aligned} \tag{6}$$

Certainly, since unsuccessful SPAC financing causes costly delay to IPO, if a project is more likely to be funded through SPAC, the entrepreneur is more willing to choose SPAC over IPO. Let $\Phi(\cdot)$ represent the CDF of the random variable $\frac{(1-\rho)\pi_{IPO}}{\pi_{SPAC}-\rho\pi_{IPO}}$, and assume that $\Phi(\cdot)$ is strictly increasing. Then at each instant, the sponsor receives good projects at the rate $\lambda p_0 \Phi(x)$.

Now we characterize the equilibrium of the SPAC game with strategic entrepreneurs. At the time $-t$, the sponsor receives bad projects at the rate $\lambda(1-p_0)$ and good projects at the rate $\lambda p_0 \Phi(x(-t))$, where

$$x(-t) \equiv \alpha_G(-t) \cdot [q + (1-q)\eta(-t)]$$

is the probability that the project can be approved if the entrepreneur chooses SPAC. Lemma 2 and Lemma 3 still hold because the projects the sponsor receives at each instant have negative NPV on average, i.e.,

$$\frac{p_0 \Phi(x(-t)) R_G + (1-p_0) R_B}{p_0 \Phi(x(-t)) + 1 - p_0} \leq p_0 R_G + (1-p_0) R_B < 1.$$

The equilibrium is very similar to that in Proposition 1 except that the sponsor receives good projects at the rate $\lambda p_0 \Phi(q + (1-q)\eta(-t))$.

Proposition 7. *As the SPAC approaches its deadline, a decreasing fraction of the entrepreneurs with good projects choose to tap the SPAC.*

As the SPAC approaches its deadline, the investor becomes more concerned about the sponsor's moral hazard problem and exerts more stringent screening. The screening effectively discourages the entrepreneurs with good projects, who can also access standard IPO, from tapping the SPAC. That means, the investor's screening abates not only the sponsor's expected payoff from proposing good projects but also the probability that he receives good projects. As analyzed in Section 4.1, this additional effect further dampens the accumulation of the sponsor's continuation value and exacerbates the sponsor's moral hazard problem. Entrepreneurs' potential strategic behavior actually stokes our concern that giving less informed investors the control right to reassure them may backfire.

5.2 Endogenous Effort to Search for Projects

In reality, the search for projects also depends on the sponsor's effort. To prepare investment proposals to the investor, the sponsor needs to spend time, energy, and money in finding projects and negotiating deals. Such effort can hardly be observed or enforced, so it is mainly determined by the sponsor's utility maximization. Since the marginal benefit of proposing a project is not constant over the lifecycle of the SPAC, he may optimally exert different amount of effort. In this subsection, we incorporate the sponsor's endogenous effort into the model.

At each instant $-t$, the sponsor can choose to exert a flow effort $\kappa(-t)$ to search for projects. It increases the arrival rate of projects from λ to $\lambda + \kappa(-t)$ without changing the probability of a good one. Meanwhile, it incurs a private flow cost $C(\kappa(-t))$ to the sponsor. $C(\cdot)$ is an increasing, convex function, and $C(0) = 0$. The equilibrium is very similar to that in Proposition 1 except that the sponsor receives projects at the rate $\lambda + \kappa^*(-t)$. $\kappa^*(-t)$ is chosen by the sponsor to maximize his continuation value, i.e.,

$$\kappa^*(-t) = \arg\max_{\kappa} \ (\lambda + \kappa)\, p_0 \cdot [F_G(-t) - V(-t)] - C(\kappa),$$

so in equilibrium it satisfies

$$C'(\kappa^*(-t)) = p_0\, [F_G(-t) - V(-t)]\,.$$

Plugging $F_G(-t) \equiv [q + (1-q)\,\eta\,(-t)]\,v_G$ into the equation, we obtain that in the second stage,

$$C'(\kappa^*(-t)) = p_0 \left[v_G q + \left(\frac{v_G}{v_B} - 1 \right) V(-t) \right]$$

and in the first stage,

$$C'(\kappa^*(-t)) = p_0 \left[v_G - V(-t) \right].$$

Proposition 8. *As the SPAC approaches its deadline, the sponsor exerts more effort in the first stage but less in the second stage.*

At every instant, the sponsor's endogenous effort is motivated by the difference between the expected payoff of proposing a good project and his continuation value. In the first stage, the benefit is always v_G, but his continuation value keeps decreasing. It implies that failing to find a good project, his situation deteriorates. Hence, he has more incentive to exert effort to search for projects. In the second stage, his situation still deteriorates, but the benefit also shrinks over time because of the investor's intensifying screening. Because a good project is more valuable than a bad one to the sponsor, $v_G > v_B$, the decrease in the sponsor's expected payoff of proposing a good project is more dramatic than that in his continuation value in an absolute basis. As a result, his incentive to exert effort is greater over time in the second stage. Similar to that on strategic entrepreneurs, this analysis also uncovers a channel through which the investor's control right may further exacerbate the sponsor's moral hazard problem.

5.3 Long-lived projects

For the baseline setup, we assume that projects are short-lived: if the sponsor doesn't propose the project he receives, the project will disappear or become unavailable immediately. With this assumption, the state of the sponsor with respect to whether he has a project and what type he has is completely independent over time. In this subsection, we explore the case of long-lived projects where the sponsor can possibly keep a project for future proposals. It turns out that such possibility does not alter the equilibrium dynamics in our setup.

The new setup is the same as the baseline one except that the projects the sponsor has received but not yet proposed still exists. Such projects are called old project. At each instant, the sponsor can choose to revisit one of the old projects. The revisit makes the project ready for proposal again at a rate of γ. It is easy to see that the sponsor must choose to revisit the best project he has received so far, so his continuation value depends on its type. Denote the sponsor's continuation value at $-t$ as $V^{\sigma}(-t)$ if the best project he has received is of the type $\sigma \in \{G, B\}$. Then in equilibrium, $V^{G}(-t) > V^{B}(-t)$. Heuristically, conditional on the game continues at time $-(t+dt)$, each instantaneous "period" $(-(t+dt), -t]$ consists of events occurring in the following order:

1. The initial state is $\sigma(-t-dt) \in \{G, B\}$;
2. With probability λdt, the sponsor receives a new project and observes its type ω', and the new project is ready for proposal;
3. With probability γdt, the best old project becomes ready;
4. If there is at least one project ready, denote the type of the best of them as $\omega(-t) \in \{G, B\}$, the sponsor proposes the best one with probability $\alpha_{\omega}(-t)$;
5. If the sponsor proposes a project, the game enters into the decision making stage and ends after that; if no project is proposed, the state is updated to $\sigma(-t)$, and the game moves on to the next period.

Since the opportunity to propose is unique, a sponsor with a project of the type ω ready for proposal faces a trade-off between $V^{\sigma}(-t)$ and $F_{\omega}(-t)$, the expected payoff of proposing it right away.

Recall that in the baseline setup, a critical observation is that the sponsor always has more incentive to propose a good project than a bad one. It follows that his expected payoff of proposing a good project is higher than that of proposing a bad one but his opportunity cost is the same for both. Although the second half does not hold in the new setup (since the sponsor's continuation value depends on the type of the projects he has received so far), we can show that this critical observation still holds.

Lemma 5. *In equilibrium, for any $-t$, $F_G(-t) > V^{G}(-t)$, so $\alpha_G(-t) = 1$.*

Notice that if the sponsor has a good project ready for proposal, his continuation value must be $V^{G}(-t)$. $F_G(-t) \leq V^{G}(-t)$ implies that the investor must have less stringent screening at some points in the future, which can compensate for the possibility that the sponsor may not have a good project ready for proposal again. However, less stringent screening increases the probability of investment in a bad project disproportionately more than that in a good one. Hence, $F_B(-t) < V^{B}(-t) < V^{G}(-t)$ must hold. Then the rest follows the proof of Lemma 2.

Proposition 9. *The new setup has a unique equilibrium, and it is the same as the one characterized by Proposition 1.*

In the baseline setup, the sponsor receives too high a fraction of bad projects so that if he proposes any bad project he receives, the investor has a negative expected profit of investing when observing M; hence, in equilibrium, the investor exerts screening that induces the sponsor to propose bad projects at only the rate

$$\lambda(1-p_0)\alpha_B(-t) = \lambda p_0 \frac{u_G - 1}{1 - u_B}.$$

In the new setup, as implied by Lemma 5, the sponsor still proposes any good project he receives right away, so revisit does not change the rate that good projects are proposed. But revisit increases the amount of bad projects ready for proposal, which makes the investor even more concerned about the average quality of proposed projects. So, the investor will exert the same screening, and the sponsor will propose bad projects at the same rate. Notably, the sponsor does not benefit from revisit. As pointed out by Proposition 2, his expected payoff depends on only the proposals of good projects in equilibrium.

5.4 Multiple agents

In the baseline setup, we assume only one investor to simplify the characterization of the equilibrium dynamics. Here we extend the model to multiple investors. If one investor's decision does not affect other investors' payoffs, then the game is essentially the same as the one-investor baseline game. However, there may be externalities between investors in practice. For a deal to be approved, it requires a sufficient fraction of investors willing to invest. In some cases, the SPAC prospectus specifies a threshold beforehand while in others, the threshold is set later to meet the minimum investment required by the project. Such a threshold allows an investor to infer information from approved investment and thus add a layer of strategic interaction between investors to the game. In this subsection, we examine the impact of this strategic interaction on the equilibrium dynamics. As shown later, the new equilibrium consists of three stages, and the new stage (the third stage) is a direct result of information aggregation through the threshold.

There are N investors in this new game. After the sponsor proposes a project, each investor observes a signal, which has the same ternary signal structure modeled in Table 1. Conditional on the type of project, the signal realizations are independent across all investors. After observing their own signals, each investor chooses to withdraw or not. There is an exogenous threshold K such that the proposal is approved if at least K investors choose not to withdraw. If the project is approved, only the investors who choose not to withdraw invest. If the project is not approved, all investors withdraw. The investors all have the same payoff structure as in the baseline setup; namely, an investor will get u_ω if she invests in a project of the type ω and get 1 if she withdraws. The sponsor's payoff is proportional to the size of the investment: if x investors invest, the sponsor receives $x \cdot v_\omega$[11].

As in the baseline setup, an investor still chooses to withdraw if he observes the signal L and not if H, so an investor's strategy is captured by the probability not to withdraw when observing M at $-t$, $\eta(-t)$. We focus on symmetric mixed-strategy equilibria where all investors have the same $\eta(-t)$ at each instant. Let $\tilde{\theta}(-t)$ be the investors' prior belief of the odds of a good project before observing any signal. Consider an investor who observes M. If he chooses not to withdraw, then he invests if and only if at least $K-1$ of the other $N-1$ investors make the same decision. Conditional on this information, the odds of a good project is

$$\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))}\tilde{\theta}(-t), \tag{7}$$

where

$$\Gamma(y) \equiv \sum_{x=K-1}^{N-1} \binom{N-1}{x} y^x (1-y)^{N-1-x}.$$

Comparing this odds to that in the baseline setup, we can see it has an additional term due to the information inferred from the threshold in equilibrium. Similar to Lemma 1, when this odds is greater (smaller) than $\frac{1-u_B}{u_G-1}$, $\eta(-t)$ is equal to 1 (0); when it is equal to $\frac{1-u_B}{u_G-1}$, $\eta(-t)$ is between 0 and 1.

Let $\tilde{\eta}(-t)$ be the sponsor's belief about the investors' strategies. By proposing a project of the type ω to the investors at $-t$, the sponsor's expected payoff is

$$F_\omega(-t) \equiv \begin{cases} v_G \cdot \Lambda(q+(1-q)\tilde{\eta}(-t)), & \text{if } \omega = G \\ v_B \cdot \Lambda((1-q)\tilde{\eta}(-t)), & \text{if } \omega = B \end{cases},$$

where

$$\Lambda(y) \equiv \sum_{x=K}^{N} \binom{N}{x} y^x (1-y)^{N-1-x} x.$$

[11] The equilibrium structure stay unchanged as long as the sponsor's payoff is weakly increasing in x.

Last, we close the model by imposing rational beliefs and D1 refinement as in Definition 1. The following proposition characterizes the unique symmetric equilibrium of the game.

Proposition 10. *The unique equilibrium of the SPAC game has three stages, and the transition time points between two consecutive stages are* $-t_1^* < -t_2^*$ *respectively.*

- *The third stage spans the period* $(-t_2^*, 0]$*, in which*
  - $\eta(-t)$ *solves*
$$\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))}\frac{p_0}{1-p_0} = \frac{1-u_B}{u_G-1};$$
  - $\alpha_G(-t) = \alpha_B(-t) = 1$*;*
  - *the sponsor's continuation value satisfies*
$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)] + \lambda(1-p_0) \cdot [F_B(-t) - V(-t)]$$
    and the boundary condition $V(0) = 0$*.*
- *The second stage spans the period* $(-t_1^*, -t_2^*]$*, in which*
  - $\eta(-t)$ *satisfies*
$$V(-t) = F_B(-t);$$
  - $\alpha_G(-t) = 1$ *and*
$$\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))}\frac{p_0}{1-p_0}\frac{\alpha_G(-t)}{\alpha_B(-t)} = \frac{1-u_B}{u_G-1};$$
  - *the sponsor's continuation value satisfies*
$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)].$$
- *The first stage spans the period* $[-T, -t_1^*)$*, in which*
  - $\eta(-t) = 1$*;*
  - $\alpha_G(-t) = 1$ *and* $\alpha_B(-t) = 0$*;*
  - *the sponsor's continuation value satisfies*
$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [v_G - V(-t)].$$

- *The transition time t_2^* satisfies $V(-t_2^*) = F_B(-t_2^*)$, the transition time t_1^* satisfies $V(-t_1^*) = v_B \cdot \Lambda((1-q))$, and $V(-t)$ is continuous at $-t_1^*$ and $-t_2^*$.*

Proposition 10 has the underlying forces similar to Proposition 1: the partial alignment of the two players' payoffs. A salient difference is the existence of the third stage in which the sponsor proposes any project he receives instantly. This cannot occur in the baseline setup because the investor will respond with the most stringent screening, $\eta(-t) = 0$, which will in turn eliminate the sponsor's desire to propose any bad project. However, with multiple investors present, an investor's confidence in the project can be bolstered by others' screening through the threshold. Note that the additional term in eq. (7), $\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))}$, goes to infinity as $\eta(-t)$ goes to 0. Even if the sponsor proposes any project he receives, the investors will choose a positive $\eta(-t)$ and thus approve a bad project with a positive probability. Although this probability could be very low, it can still induce the sponsor to propose bad projects near the deadline.

6 Concluding Remark

Studying SPAC from a perspective of delegated investment, this paper focuses on the the strategic interaction between a sponsor and a investor. Consistent with the conventional wisdom, the incentive misalignment of the two parties gives rise to a moral hazard problem of the sponsor. However, this is not the whole story. The alignment side of the two parties' incentives helps mitigate the problem. A key takeaway is that giving less informed investors much control right may exacerbate the moral hazard problem and make everyone worse off.

SPAC in reality is a complicated business that involves many parties and interactions. To better illustrate our main idea, we abstract away from several elements of SPAC. Here are some that we think are important and merit more research.

1. The secondary market of SPAC shares. SPAC shares are publicly traded, which aggregates investors' information and affects their decisions.
2. Private Investment in Public Equity (PIPE). An SPAC sponsor frequently invites a PIPE investment as a part of the business combination, which further complicates their incentives.
3. Investors' demand for liquidity. Besides profitability, liquidity is another critical reason why some investors favor SPACs. Potentially, concern or demand for liquidity may affect investors' decisions as well.

References

Aghion, P. and J. Tirole (1997). Formal and real authority in organizations. *Journal of Political Economy 105*(1), 1–29.

Bai, J., A. Ma, and M. Zheng (2021). Segmented Going-Public Markets and the Demand for SPACs 1. *Working Paper*.

Blomkvist, M. and M. Vulanovic (2020). SPAC IPO waves. *Economics Letters 197*, 109645.

Chatterjee, S., N. K. Chidambaran, and G. Goswami (2016). Security design for a non-standard IPO: The case of SPACs. *Journal of International Money and Finance 69*, 151–178.

Crawford, B. Y. V. P. and J. Sobeli (1982). Strategic Information Transmission. *Econometrica 50*(6), 1431–1451.

Cumming, D., L. H. Haß, and D. Schweizer (2014). The fast track IPO - Success factors for taking firms public with SPACs. *Journal of Banking and Finance 47*(1), 198–213.

Dessein, W. (2002). Authority and communication in organizations. *Review of Economic Studies 69*(4), 811–838.

Dimitrova, L. (2017). Perverse incentives of special purpose acquisition companies, the "poor man's private equity funds". *Journal of Accounting and Economics 63*(1), 99–120.

Gahng, M., J. R. Ritter, and D. Zhang (2021). SPACs. *Working Paper* (352).

Grenadier, S. R., A. Malenko, and N. Malenko (2016). Timing Decisions in Organizations : Communication and Authority in a Dynamic Environment. *American Economic Review 106*(9), 2552–2581.

Guo, Y. (2016). Dynamic delegation of experimentation. *American Economic Review 106*(8), 1969–2008.

Klausner, M., M. Ohlrogge, and E. Ruan (2020). A Sober Look at SPACs. *Working Paper*, 1–57.

Shachmurove, Y. and M. Vulanovic (2017). SPAC IPOs. *Working Paper*, 1–40.

A Proofs

Proof of Lemma 2

Suppose $F_G(-t) \leq V(-t)$ at a time $-t$. Then $F_B(-t) < V(-t)$, so $\alpha_B(t) = 0$. If $\alpha_G(t) > 0$, by the investor's rational belief in equilibrium, $\tilde{\theta}(-t) = +\infty$. According to Lemma 1, $\eta(-t) = 1$, so $F_G(-t) = v_G$. Since looking forward in the future, the sponsor always expects a positive probability of no investment, $V(-t) < v_G = F_G(-t)$. Contradiction! If $\alpha_G(t) = 0$, by D1 refinement, $\tilde{\theta}(-t) = +\infty$. Following the same argument, we encounter contradiction. Finally, $F_G(-t) > V(-t)$ directly implies $\alpha_G(-t) = 1$ and strictly decreasing continuation value over time.

Proof of Lemma 3

Suppose $V(-t) < v_B(1-q)$ and $t > 0$. If $\alpha_B(-t) = 0$, following the proof of Lemma 2, we have $\eta(-t) = 1$, so $F_B(-t) = (1-q)v_B > V(-t)$. Contradiction! If $\alpha_B(-t) = 1$,

$$\tilde{\theta}(-t) \leq \frac{p_0}{1-p_0} < \frac{1-R_B}{R_G-1} < \frac{1-u_B}{u_G-1},$$

which implies $\eta(-t) = 0$ according to Lemma 1. Then $F_B(-t) = 0 < V(-t)$. Contradiction! Therefore, $\alpha_B(-t) \in (0,1)$. $V(-t) > (1-q)v_B$ implies $V(-t) > F_B(-t)$, so $\alpha_B(-t) = 0$.

Proof of Proposition 3

For $t < t^*$,

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)].$$

Plugging $F_G(-t) \equiv q + (1-q)\eta(-t)$ and the equilibrium condition $V(-t) = (1-q)\eta(-t)v_B$ into the equation, we obtain

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot \left[\left(\frac{v_G}{v_B} - 1\right)V(-t) + qv_G\right].$$

Combining with the boundary condition $V(0) = 0$, we obtain

$$V(-t) = \left[e^{\lambda p_0 \left(\frac{v_G}{v_B} - 1\right) t} - 1 \right] \frac{1}{\frac{v_G}{v_B} - 1} \cdot q \cdot v_G.$$

So, t^* satisfies eq. (4).

Proof of Lemma 4

In the second stage where $-t > -t_s^*$,

- the sponsor's continuation value $V_s(-t)$ satisfies

$$\frac{dV_s(-t)}{dt} = \lambda \left[p_0 v_G + (1 - p_0) v_B - V_s(-t) \right]$$

  and two boundary conditions $V_s(0) = 0$ and $V_s(-t_s^*) = v_B$;
- the investor's continuation value $U_s(-t)$ satisfies

$$\frac{dU_s(-t)}{dt} = \lambda \cdot \left[p_0 u_G + (1 - p_0) u_B - U_s(-t) \right]$$

  and one boundary condition $U_s(0) = 1$.

In the first stage where $-t < -t_s^*$,

- $V_s(-t)$ satisfies

$$\frac{dV_s(-t)}{dt} = \lambda p_0 \cdot \left[v_G - V_s(-t) \right]$$

  and one boundary condition $V_s(-t_s^*) = v_B$;
- $U_s(t)$ satisfies

$$\frac{dU_s(-t)}{dt} = \lambda p_0 \cdot \left[u_G - U_s(-t) \right]$$

  and one boundary condition that requires $U_s(-t)$ is continuous at $-t_s^*$.

According to the evolution of the sponsor's continuation value, we obtain that for $-t \geq -t_s^*$,

$$V_s(-t) = \left(1 - e^{-\lambda t}\right) \left[p_0 v_G + (1 - p_0) v_B \right].$$

The first point is directly implied by the boundary condition $V_s(-t_s^*) = v_B$.

According to the evolution of the investor's continuation value, we obtain that for $-t \geq -t_s^*$,

$$U_s(t) = \left(1 - e^{-\lambda t}\right)[p_0 u_G + (1-p_0)u_B] + e^{-\lambda t}$$

and for $-t < -t_s^*$,

$$U_s(-t) = \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t - t_s^*)} U_s(-t_s^*).$$

Notice that $U_s(-t) < 1$ for $t \geq -t_s^*$ and $U_s(-t)$ increases to u_G as t increases from t_s^* to $+\infty$. T_s^* exists, and $T_s^* > t_s^*$.

Proof of Proposition 4

Here we prove the second point. Since $T > T_s^*$, $T > t_s^*$.

First, since $U_s(-t_s^*) < 1$,

$$\begin{aligned} U_s(-T) &= \left(1 - e^{-\lambda p_0 (T - t_s^*)}\right) u_G + e^{-\lambda p_0 (T - t_s^*)} U_s(-t_s^*) \\ &< \left(1 - e^{-\lambda p_0 (T - t_s^*)}\right) u_G + e^{-\lambda p_0 (T - t_s^*)} \\ &< \left(1 - e^{-\lambda p_0 T}\right) u_G + e^{-\lambda p_0 T}. \end{aligned}$$

Second, $U(-T)$ is strictly increasing in q. According to Proposition 3, t^* is strictly decreasing in q. If $t^* \geq T$,

$$U(-T) = \left(1 - e^{-\lambda p_0 \frac{u_G - u_B}{1 - u_B} T}\right) \frac{1 - u_B}{u_G - u_B} q \cdot (u_G - 1) + 1,$$

which is strictly increasing in q.

Suppose $t^* < T$. Consider $\tilde{q}$ marginally smaller than q such that its corresponding $\tilde{t}^*$ is smaller than T as well. To reflect different q, we write the investor's continuation value as $U(-t; q)$. Since $0 < t^* < \tilde{t}^*$,

$$\begin{aligned} U(-T; q) &= \left(1 - e^{-\lambda p_0 (T - t^*)}\right) u_G + e^{-\lambda p_0 (T - t^*)} U(-t^*; q) \\ &= \left(1 - e^{-\lambda p_0 (T - \tilde{t}^*)}\right) u_G + e^{-\lambda p_0 (T - \tilde{t}^*)} U(-\tilde{t}^*; q) \end{aligned}$$

We just need to show $U(-\tilde{t}^*;q) > U(-\tilde{t}^*;\tilde{q})$. Denote $e^{\lambda p_0(\tilde{t}^*-t^*)}$ as a and $\frac{u_G-u_B}{1-u_B}$ as x.

$$\begin{aligned}
U(-\tilde{t}^*;\tilde{q})-1 &= \left(1-e^{-\lambda p_0\frac{u_G-u_B}{1-u_B}\tilde{t}^*}\right)\frac{1}{\frac{u_G-u_B}{1-u_B}}\tilde{q}\cdot(u_G-1)\\
&= \left(1-e^{-\lambda p_0\frac{u_G-u_B}{1-u_B}(\tilde{t}^*-t^*)}\right)\frac{1}{\frac{u_G-u_B}{1-u_B}}\tilde{q}\cdot(u_G-1)+e^{-\lambda p_0\frac{u_G-u_B}{1-u_B}(\tilde{t}^*-t^*)}\left(U(-t^*;\tilde{q})-1\right)\\
&= \left(1-a^{-x}\right)\frac{\tilde{q}\cdot(u_G-1)}{x}+a^{-x}\left(U(-t^*;\tilde{q})-1\right).
\end{aligned}$$

Since $a>1$ and $x>1$, $a^{-x}<a^{-1}$ and

$$\frac{1-a^{-x}}{x}<1-a^{-1}.$$

So,

$$\begin{aligned}
U(-\tilde{t}^*;\tilde{q})-1 &< \left(1-a^{-1}\right)\tilde{q}\cdot(u_G-1)+a^{-1}\left(U(-t^*;\tilde{q})-1\right)\\
&< \left(1-a^{-1}\right)(u_G-1)+a^{-1}\left(U(-t^*;q)-1\right)\\
&= U(-\tilde{t}^*;q)-1.
\end{aligned}$$

As $q\to 0$, $t^*\to+\infty$, so

$$U(-T)\to\lim_{q\to 0}\left[\left(1-e^{-\lambda p_0\frac{u_G-u_B}{1-u_B}T}\right)\frac{1-u_B}{u_G-u_B}q\cdot(u_G-1)+1\right]=1.$$

As $q\to 1$, $t^*\to 0$, so

$$U(-T;q)\to\lim_{q\to 1}\left[\left(1-e^{\lambda p_0(t^*-T)}\right)u_G+e^{\lambda p_0(t^*-T)}U(-t^*;q)\right]=\left(1-e^{-\lambda p_0 T}\right)u_G+e^{-\lambda p_0 T}.$$

We obtain the second point.

Proof of Proposition 5

In the second stage where $-t>-t_c^*$,

- $V_c(-t)$ in this stage satisfies

$$\frac{dV_c(-t)}{dt} = \lambda\left[p_0 \cdot v_G + (1-p_0)\cdot(1-\hat{q})v_B - V_c(-t)\right]$$

  and two boundary conditions $V_c(0) = 0$ and $V_c(-t_c^*) = (1-\hat{q})v_B$;
- $U_c(-t)$ satisfies

$$\frac{dU_c(-t)}{dt} = \lambda \cdot \left[p_0 u_G + (1-p_0)(1-\hat{q})u_B + (1-p_0)\hat{q} - U_c(-t)\right]$$

  and one boundary condition $U_c(0) = 1$.

In the first stage where $-t < -t_c^*$,

- $V_c(-t)$ in this stage satisfies

$$\frac{dV_c(-t)}{dt} = \lambda p_0 \cdot \left[v_G - V_c(-t)\right]$$

  and one boundary condition $V_c(-t_c^*) = (1-\hat{q})v_B$.
- $U_c(-t)$ satisfies

$$\frac{dU_c(-t)}{dt} = \lambda p_0 \cdot \left[u_G - U_c(-t)\right]$$

  and one boundary condition that requires $U_c(-t)$ is continuous at $-t = -t_c^*$.

First, $t_c^* < t_s^*$. They satisfy respectively

$$\left(1 - e^{-\lambda t_s^*}\right)\left[p_0 v_G + (1-p_0)v_B\right] = v_B$$
$$\left(1 - e^{-\lambda t_c^*}\right)\left[p_0 v_G + (1-p_0)(1-\hat{q})v_B\right] = (1-\hat{q})v_B.$$

So,

$$1 - e^{-\lambda t_s^*} > 1 - e^{-\lambda t_c^*} \Leftrightarrow t_s^* > t_c^*.$$

For $t \in (0, t_c^*]$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda t}\right)[p_0 u_G + (1-p_0)(1-\hat{q})u_B + (1-p_0)\hat{q}] + e^{-\lambda t} \\ &> \left(1 - e^{-\lambda t}\right)[p_0 u_G + (1-p_0)u_B] + e^{-\lambda t} \\ &= U_s(-t). \end{aligned}$$

For $t \in (t_c^*, t_s^*]$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda p_0 (t - t_c^*)}\right) u_G + e^{-\lambda p_0 (t - t_c^*)} U_c(-t_c^*) \\ &> \left(1 - e^{-\lambda p_0 (t - t_c^*)}\right)[p_0 u_G + (1-p_0)u_B] + e^{-\lambda p_0 (t - t_c^*)} U_s(-t_c^*) \\ &= U_s(-t). \end{aligned}$$

For $t \in (t_s^*, +\infty)$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t - t_s^*)} U_c(-t_s^*) \\ &> \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t - t_s^*)} U_s(-t_s^*) \\ &= U_s(-t). \end{aligned}$$

Therefore, $U_c(-T) > U_s(-T)$.

Proof of Proposition 6

In the second stage where $-t > -t_v^*$,

- $V_v(-t)$ satisfies
$$\frac{dV_v(-t)}{dt} = \lambda p_0 \cdot q\,[v_G - V_v(-t)]$$
  and two boundary conditions $V_v(0) = 0$ and $V_v(-t_v^*) = v_B$.;
- $U_v(-t)$ satisfies
$$\frac{dU_v(-t)}{dt} = \lambda p_0 \cdot q\,[u_G - U_v(-t)]$$
  and one boundary condition $U_v(0) = 1$.

In the first stage where $-t < -t_v^*$,

- $V_v(-t)$ satisfies
$$\frac{dV_v(-t)}{dt} = \lambda p_0 \cdot [v_G - V_v(-t)]$$
and one boundary condition $V_v(-t_v^*) = v_B$;
- $U_v(t)$ satisfies
$$\frac{dU_v(-t)}{dt} = \lambda p_0 [u_G - U_v(-t)]$$
and one boundary condition that requires $U_v(-t)$ is continuous at $-t_v^*$.

First, for $-t > \max\{-t^*, -t_v^*\}$, $V_v(-t) < V(-t)$.

$$V(-t) = \left[e^{\lambda p_0\left(\frac{v_G}{v_B}-1\right)t} - 1\right] \frac{1}{\frac{v_G}{v_B} - 1} \cdot q \cdot v_G$$
$$V_v(-t) = \left(1 - e^{-\lambda p_0 qt}\right) v_G.$$

Note that for $a > 1$, $\frac{a^x-1}{x}$ is increasing in x and $\frac{1-a^{-x}}{x}$ is decreasing in x. Since $\frac{v_G}{v_B} - 1 > 0$,

$$\begin{aligned} V(-t) &> \lim_{x\downarrow 0} \left[e^{\lambda p_0 t \cdot x} - 1\right] \frac{1}{x} \cdot q \cdot v_G \\ &= \lambda p_0 t \cdot q \cdot v_G. \end{aligned}$$

On the other hand,

$$\begin{aligned} V_v(-t) &= \frac{1 - e^{-\lambda p_0 qt}}{\lambda p_0 qt} \lambda p_0 qt v_G \\ &< \lim_{x\downarrow 0} \frac{1 - e^{-x}}{x} \lambda p_0 qt v_G \\ &= \lambda p_0 qt v_G. \end{aligned}$$

So, $V_v(-t) < V(-t)$.

Second, $t^* < t_v^*$. They satisfy respectively

$$V(-t^*) = (1-q)v_B$$
$$V_v(-t_v^*) = v_B.$$

$V(-t^*) < V_v(-t_v^*)$ implies $t^* < t_v^*$.

For $t \in (0, t^*]$, obviously $V(-t) > V_v(-t)$.

For $t \in (t^*, t_v^*]$, since $v_G > V(-t^*) > V_v(-t^*)$,

$$\begin{aligned} V(-t) &= \left(1 - e^{-\lambda p_0 (t-t^*)}\right) v_G + e^{-\lambda p_0 (t-t^*)} V(-t^*) \\ &> \left(1 - e^{-\lambda p_0 q (t-t^*)}\right) v_G + e^{-\lambda p_0 q (t-t^*)} V(-t^*) \\ &> \left(1 - e^{-\lambda p_0 q (t-t^*)}\right) v_G + e^{-\lambda p_0 q (t-t^*)} V_v(-t_c^*) \\ &= V_v(-t). \end{aligned}$$

For $t \in (t_v^*, +\infty)$,

$$\begin{aligned} V(-t) &= \left(1 - e^{-\lambda p_0 (t-t_v^*)}\right) v_G + e^{-\lambda p_0 (t-t_v^*)} V(-t_v^*) \\ &> \left(1 - e^{-\lambda p_0 (t-t_v^*)}\right) v_G + e^{-\lambda p_0 (t-t_v^*)} V_v(-t_v^*) \\ &= V_v(-t). \end{aligned}$$

Therefore, $V(-T) > V_v(-T)$.

Proof of Lemma 5

Consider any $-t$ and suppose the type of the best project the sponsor has received until that is σ. The sponsor's proposal strategy $(\alpha_\omega(\cdot))_{\omega \in \{G,B\}}$[12] implies a pair of functions $(f_\omega(\cdot))_{\omega \in \{G,B\}}$: $f_\omega(-\tau)$ represents the unconditional probability density that the sponsor proposes a project of the type ω at $-\tau$. Accordingly, the sponsor's expected payoff by adopting this strategy is

$$\tilde{V}(-t, f_G, f_B) \equiv \int_0^t F_G(-\tau) f_G(-\tau) d\tau \cdot v_G + \int_0^t F_B(-\tau) f_B(-\tau) d\tau \cdot v_B.$$

Specifically, denote the densities resulting from the sponsor's optimal proposal strategy as f_G^σ and f_B^σ respectively. Then

$$V^\sigma(-t) = \tilde{V}(-t, f_G^\sigma, f_B^\sigma).$$

[12] Note that in the new setup, the sponsor's strategy is also based on the type of the best project he has received until then besides the time $-t$.

Next, suppose $F_G(-t) \leq V^G(-t)$. Then

$$[q+(1-q)\eta(-t)] \cdot v_G \leq \int_0^t [q+(1-q)\eta(-\tau)] f_G^G(-\tau) d\tau \cdot v_G + \int_0^t (1-q)\eta(-\tau) f_B^G(-\tau) d\tau \cdot v_B.$$

Since $v_G > v_B$ and

$$q+(1-q)\eta(-\tau) \geq \eta(-\tau) \geq (1-q)\eta(-\tau),$$

it further implies

$$q+(1-q)\eta(-t) \leq \int_0^t [q+(1-q)\eta(-\tau)] \left[f_G^G(-\tau) + f_B^G(-\tau) \right] d\tau.$$

Since it is always possible that the sponsor may not have any project ready for proposal in the future,

$$\int_0^t \left[f_G^G(-\tau) + f_B^G(-\tau) \right] d\tau < 1.$$

Hence,

$$\eta(-t) < \int_0^t \eta(-\tau) \left[f_G^G(-\tau) + f_B^G(-\tau) \right] d\tau.$$

We claim that $F_B(-t) < V^B(-t)$ must hold. Consider the sponsor with $\sigma = B$ at $-t$. Imagine that he mistakenly regards one of his old projects as good, always revisits it, and plays the optimal proposal strategy of the sponsor with $\sigma = G$ at $-t$. Let f_G and f_B represent the the true unconditional probability densities implied by this strategy. The sponsor thinks he will end up with the unconditional probability densities f_G^σ and f_B^σ, but some "good" projects he proposes are actually bad. Therefore, for any $-\tau \in (-t, 0]$,

$$f_G(-\tau) + f_B(-\tau) = f_G^G(-\tau) + f_B^G(-\tau),$$
$$f_G(-\tau) \leq f_G^G(-\tau).$$

Note that the sponsor's optimal strategy should be no worse than this mimicking strategy. So,

$$\begin{aligned}V^B(-t) &\geq \int_0^t [q+(1-q)\eta(-\tau)]\, f_G(-\tau)d\tau \cdot v_G + \int_0^t (1-q)\eta(-\tau)\, f_B(-\tau)d\tau \cdot v_B \\ &\geq \int_0^t (1-q)\eta(-\tau)\, f_G(-\tau)d\tau \cdot v_B + \int_0^t (1-q)\eta(-\tau)\, f_B(-\tau)d\tau \cdot v_B \\ &= (1-q)\cdot v_B \cdot \int_0^t \eta(-\tau)\left[f_G^G(-\tau)+f_B^G(-\tau)\right]d\tau \\ &> (1-q)\eta(-t)\cdot v_B = F_B(-t).\end{aligned}$$

Following the proof of Lemma 2, we will encounter contradiction. So, $F_G(-t) > V^G(-t)$. and $\alpha_G(-t) = 1$.

Proof of Proposition 9

First, $V^B(-t)$ strictly decreases to 0 as $-t$ increases to 0 because

$$\frac{dV^B(-t)}{dt} \geq \lambda p_0 \cdot \left[F_G(-t) - V^B(-t)\right] > 0.$$

Second, following the logic similar to Lemma 3, we obtain that when $V^B(-t) < (1-q)\, v_B$ and $t > 0$, $\alpha_B(-t) \in (0,1)$; when $V^B(-t) > (1-q)\, v_B$, $\alpha_B(-t) = 0$. Combining the two, we obtain a unique equilibrium of the game.