Reputation Concerns Under At-Will Employment*

JIAN SUN† DONG WEI‡

MIT UC Berkeley

April 30, 2019

[Preliminary Draft]

Abstract

We study a continuous-time model of long-run employment relationship with fixed wage and at-will firing; that is, termination of the relationship is non-contractible. Depending on his type, the worker either always works hard, or can freely choose his effort level. The firm does not know the worker's type and the monitoring is imperfect. We show that, in the unique Markov equilibrium, as the worker's reputation worsens, his job becomes less secure and the strategic worker works harder. We further demonstrate that the relationship between average productivity and job insecurity is U-shaped, which is consistent with typical findings in the organizational psychology literature.

Keywords: moral hazard, job insecurity, reputation, at-will employment

JEL classification: C73, D83, J24, M51

*We thank Haluk Ergin, Brett Green, Elliot Lipnowski, Andrey Malenko, Harry Pei, Philipp Strack, and Haoxiang Zhu for helpful discussions and comments. All errors are our own.

†jiansun@mit.edu

‡dong.wei@berkeley.edu

1 Introduction

Employment relationships are usually subject to imperfect monitoring. Whereas an employer can observe certain indicators of a worker's performance (such as sale statistics of a salesperson, number of cases processed by a bureaucrat, etc.), the stochastic components of such indicators make it hard to tell a worker's actual effort.

Meanwhile, an employer has at least two tools to regulate the incentives of a worker: wages and the threat to fire the worker. In some countries, in particular the United States, termination of an employment relationship is "at will", where a worker can be fired for virtually any or no reason.[1] As the Supreme Court of California explained,

> *[A]n employer may terminate its employees at will, for any or no reason ... the employer may act peremptorily, arbitrarily, or inconsistently, without providing specific protections such as prior warning, fair procedures, objective evaluation, or preferential reassignment ... The mere existence of an employment relationship affords no expectation, protectible by law, that employment will continue, or will end only on certain conditions, unless the parties have actually adopted such terms.*

Although many studies have been dedicated to understanding optimal wage contracts (Holmstrröm 1979; Sannikov 2008; Edmans et al. 2012, etc.), relatively few is focused on the role of such non-contractible firing threats.[2]

In this paper, in a fixed-wage environment, we study how a firm's at-will firing shapes a worker's incentive to shirk when the latter has reputation concerns. In our model, a long-run firm is dealing with one of the two possible types of long-run workers: the good type (G) who always works hard, and the bad type (B) who can freely choose how much to shirk. While the worker's type and past actions are his private information, the cumulative output process, whose drift is determined by the worker's effort, is commonly observed; such a history helps the firm form beliefs about the worker's type and predict later actions. At each instant, the firm chooses whether to continue employing the worker, and the worker, if still employed, decides how much effort to exert. Compared to its outside option, the employment relationship is profitable to the firm if and only if the worker is of the good type (who never shirks) or he is of the bad type but does not shirk too much.

We focus on Markov strategies where each player's action only depends on the firm's current belief about the worker's type, which we interpret as the worker's reputation.[3] Under a refinement

[1] Exceptions include: firing because of the employee's race or religion, termination that violates the "public policy doctrine" (e.g. retaliating the employee for performing an action that complies with public policy), etc.

[2] For an exception, see Kuvalekar and Lipnowski (2018).

[3] Although the worker's type and effort are his private information, the *firm's* belief, which only depends the real-

introduced by Kuvalekar and Lipnowski (2018), we characterize the unique Markov equilibrium of this game for all relevant parameter values. In such a equilibrium, the firm retains the worker for sure when the worker's reputation for being a good type is high enough. As the worker's reputation worsens, the firm increases its firing probability until its belief reaches a point after which it fires the worker for sure.[4] Meanwhile, the bad worker shirks to the maximum level (i.e., exerts least effort) when the firm trusts him sufficiently; as firing becomes imminent, the bad worker starts to be more cautious and work harder. See Figure 1 for an illustration.

To understand their behavior, note that in our model, the bad worker can at best work as hard as the good worker, so that the firm always weakly prefers to hire a good type. As the probability of the bad type increases, the firm becomes more inclined to fire the worker. For the bad worker, shirking increases his instantaneous payoff, while it also speeds up the firm's learning of his type and thus increases the probability of getting unemployed. When deciding how much to shirk, he needs to balance its marginal benefit and cost. When the firm sufficiently trusts him, it is quite unlikely for him to get fired in the near future; as a result, his incentive to shirk is very high, because it saves the effort cost while not affecting much the firing probability. As the firm becomes more suspicious, the worker feels more urgent to slow down the firm's learning, and thus works harder.

We also examine how average productivity (from the firm's viewpoint) varies with job insecurity, measured by the firm's belief about the probability of the *bad* worker. We show that the relationship is always U-shaped; that is, average productivity first decreases and then increases with job insecurity (see Figure 3). Intuitively, when his job is very secure, the bad worker constantly exerts minimum effort; in this region, if the belief about the bad worker increases, the average productivity decreases accordingly. As the job becomes increasingly insecure, we explained before that the bad worker starts to worker harder; it turns out that the increase in the bad worker's effort eventually outweighs the worsening of reputation, so that the average productivity will start to increase. This result is consistent with mixed findings in the organizational psychology literature, which show that the effect of job insecurity on productivity is usually ambiguous: it is sometimes found to be positive (Probst et al., 2007; Sverke and Hellgren, 2001), negative (Reisel et al., 2010; Roskies and Louis-Guerin, 1990), and in particular, U-shaped (Selenko et al., 2013).

Related Literature. This paper examines the reputation dynamics in a fixed-wage employment relationship with at-will firing. It fits into the literature on dynamic games with reputation, initiated by Kreps and Wilson (1982) and Milgrom and Roberts (1982). Many classical studies focus on characterizing equilibrium payoffs. Most notably, general analyses of the discrete-time setup with one sufficiently patient long-run player deliver sharp bounds on players' equilibrium

ized output process and its *conjecture* about the worker's action, is always common knowledge, both on and off the equilibrium path.

[4] The middle region where the firm fires the worker stochastically may not exist.

payoffs (Fudenberg and Levine, 1989, 1992). Atakan and Ekmekci (2012, 2013) extend previous analyses to the cases with two patient long-run players and possibly two-sided incomplete information. Faingold and Sannikov (2011) introduce the continuous-time analogue of Fudenberg and Levine (1992)'s setting (a long-run player v.s. a population of short-run players), and obtain a clean characterization of sequential equilibria for fixed discount rates, using ordinary differential equations. A novel element of our model is that the firm's action is a stopping time. Moreover, in a specific context of employment relationship featuring two long-run players, we are able to analytically solve both players' equilibrium actions and payoffs. Halac and Prat (2016) study a model of employment relationship where the *firm* has an endogenous type that depends on the current attention technology to recognize (and thus reward) the worker's good performance. They show that, if the attention technology can occasionally break down, the firm's incentive to costly restore the technology is not sufficient to stop the decline in worker's effort over time, which leads to the deterioration of the firm's reputation and the value of their relationship.

Another related literature examines "career-concerns" type of models started by Holmström (1999). Distinct from our setup and the reputation games in general, an important assumption in these models is symmetric uncertainty between players about the underlying state (e.g., quality of a worker, difficulty level of a task, etc.). That is, along the equilibrium path both players have the *same* information about the underlying state. Models with both symmetric uncertainty and private actions are notably hard to analyze, because when the player with private actions deviates from the equilibrium play, its belief about the underlying state will be different from that of the other player, and thus becomes its private information. Recent works, such as Cisternas (2017) and Bhaskar and Mailath (2019), are among the first attempts to address this issue. The most related paper to ours is Kuvalekar and Lipnowski (2018), who look at a similar environment under the assumption of symmetric uncertainty about match quality as well as *costless* and *observable* effort. Though the equilibrium structures in these two models resemble each other, the economic forces generating such predictions, as well as its implication on firm's learning speed, are very different. We delay a more detailed discussion on the relation between these two papers to the end of Section 3.

The rest of the paper is organized as follows. Section 2 sets up the model. Sections 3 presents our main results on the equilibrium characterization. Section 4 discusses several extensions, including costly worker replacement and flexible wage. Section 5 concludes.

2 Model

2.1 Model Setup

Suppose that two agents enter a long-term relationship in continuous time which runs from 0 to ∞. To fix ideas, we call them firm ("it") and worker ("he"). At each instant, the firm decides whether or not to irreversibly fire the worker, while the worker, depending on his type, may take an action that affects both parties' payoffs. Wage is fixed, and neither party has commitment power.

The worker can be one of the two types, $\theta \in \{0, 1\}$, where we refer to $\theta = 0$ as the good type (G) and $\theta = 1$ as the bad type (B). Type G is hardworking and never shirks. Type B, on the other hand, is aware of the option to shirk (divert money) which benefits himself at the cost of the firm, and he decides how much to shirk at each instant. The worker perfectly knows his own type, while the firm can only gradually learn it from a stochastic output process whose mean is determined by the worker's action.

Payoffs

We model the firm's profit and worker's income as diffusion processes whose drifts depend on the worker's type and action. Let Π_t and W_t be the firm's cumulative profit and the worker's cumulative income, respectively, up to time t. The laws of motion for these processes are:

$$\begin{aligned} d\Pi_t &= (y - \theta a_t)dt + \sigma dB_t, \\ dW_t &= (w + \theta a_t)dt, \end{aligned}$$

where $y, w > 0$ are some fixed instantaneous payoffs this relationship generates to these two parties, and the worker's action a_t from $[0, \bar{a}]$ can be interpreted as how much he shirks or how much money he diverts at time t. During time period $[t, t + dt)$, the firm's accrued profit is $d\Pi_t$ if it employs the worker, and the worker's accrued income is dW_t. In this model, $y + w$ is the total surplus of the relationship at each instant which is not affected by the worker's type or action; his action only determines how the surplus is split. Type G can be understood as always choosing $a = 0$ (i.e., an commitment type), as his action does not affect any party's payoff; *thus, from now on, we use a_t to denote type B's action at time t and focus on type B's strategy.*

The game ends after the worker is fired, and we normalize both players' outside values to zero. Therefore, given a (stochastic) firing time τ, the realized payoffs of the firm and the worker, respectively, are:

$$\int_0^\tau e^{-rt} d\Pi_t \text{ and } \int_0^\tau e^{-rt} dW_t,$$

where r is the common discount factor.

We assume that $w > 0$ and $0 < y < \bar{a}$. This means that the worker always wants to stay employed regardless of the action he takes, while the firm incurs a loss if the worker is of type B and shirks too much. By counting time and payoffs in appropriate units, we can without loss of generality normalize $\bar{a} = \sigma = 1$.[5]

Information and Learning

We assume that the firm's profit process Π_t is publicly observable, while the worker's type θ, action a_t and income process W_t are his private information. The uncertainty about the worker's type is asymmetric: the worker perfectly knows it, while the firm starts with an initial belief $\mathbb{E}_0(\theta) = p_0 \in (0, 1)$, and can gradually update his belief about θ based on the observed Π_t using Bayes' rule. Since the profit process is public, at each t the firm's belief p_t is common knowledge.

At time t, the information set of the firm contains $(\Pi_{\tilde{t}})_{\tilde{t}<t}$ while that of the worker contains $(\theta, a_{\tilde{t}}, \Pi_{\tilde{t}})_{\tilde{t}<t}$.[6] Since actions are not observable to the firm, the firm's interpretation of an observed profit process depends on its *conjecture* about the worker's action. Similar to the derivation in Bolton and Harris (1999), if the firm's conjecture about type B worker's action at time t is $\tilde{a}_t$ while the observed output is $d\Pi_t$, then the law of motion of the belief process is

$$dp_t = -\tilde{a}_t p_t (1 - p_t) \left[d\Pi_t - (y - \tilde{a}_t p_t) dt \right]. \tag{1}$$

Since the firm's information set is contained in that of the worker, for any *given realization* of $d\Pi_t$, their perceptions of the firm's belief p_t always coincide.[7] Meanwhile, the stochastic process p_t is different across their respective viewpoints because the *distributions* of $d\Pi_t$ in (1) are different for these two players. From the firm's viewpoint, $d\Pi_t - (y - \tilde{a}_t p_t)dt$ is a Brownian motion and p_t is a martingale. From the (type B) worker's viewpoint, since he also knows θ and a_t, we have[8]

$$dp_t = \tilde{a}_t p_t (1 - p_t) \left[(a_t - p_t \tilde{a}_t) dt - dB_t \right]. \tag{2}$$

As a result, from the worker's viewpoint p_t is *not* a martingale.

We interpret p_t as the worker's reputation. Because p_t is the probability of the bad worker, *we say that the worker has a good/better reputation if p_t is small/smaller, and that he has a bad/worse*

[5] This normalization implicitly makes $y < 1$.

[6] Note that the W_t process is fully determined by the a_t process. So given a history of actions, there is no new information contained in the corresponding history of income.

[7] Even though actions are not observable to the firm and the worker may take different actions from what the firm expects, the *firm's* belief about θ, which only depends on p_0 and Π_t, is always common knowledge. Hence, under asymmetric uncertainty, off-equilibrium actions do *not* create additional complications to belief updating even if action histories are the worker's private information.

[8] The worker can back out the random term dB_t in the output process from his information set $\{d\Pi_t, a_t, \theta\}_t$ by $dB_t = d\Pi_t - (y - \theta a_t)dt$. The firm cannot back out dB_t as it only has information on $d\Pi_t$.

reputation if p_t is large/larger.

2.2 Player Strategies

At each t and each information set, the worker chooses an action $a_t \in [0,1]$, and the firm chooses a termination rate $s_t \in [0,\infty]$ under which the relationship is terminated with probability $s_t dt$ during $[t, t+dt)$.[9] The relationship is terminated with certainty at time t if $s_t = \infty$.

We focus on Markov strategies which use the worker's reputation as state variable. Markov strategies are appealing due to its simplicity, and are widely used in the literature (for example, see Mailath and Samuelson 2001; Faingold and Sannikov 2011, etc.). Indeed, both players summarize the information contained in the entire output history into the worker's reputation and base their decisions on such a variable.

Specifically, let $\bar{\mathcal{A}}$ be the set of measurable mappings from $(0,1)$ to $[0,1]$, s.t. for all $a, \tilde{a} \in \bar{\mathcal{A}}$, both (1) and (2) admit a unique weak solution.[10] We define Markov strategies as follows.

Definition 1. *A* ***Markov strategy profile*** *is a pair (a, s), such that*

1. *$a : (0,1) \to [0,1]$ is piecewise Lipschitz with no removable discontinuities,*[11] *and $a \in \bar{\mathcal{A}}$;*
2. *$s : (0,1) \to [0,\infty]$ is piecewise continuous with no removable discontinuities; it is piecewise Lipschitz on any interval over which s is bounded, and $s^{-1}(\infty)$ is a closed set.*

Let $\mathcal{A}$ and $\mathcal{S}$ be the sets of strategies specified above for the worker and the firm, respectively.

Recall that the worker of type G always takes action 0, so it is enough to specify the strategy of the type B worker using function a. We say that two strategies are identical if they agree almost everywhere. In this definition, discontinuities are allowed in players' strategies, while to be able to solve the model, we assume enough smoothness on intervals where they are continuous.

[9] In this model, the worker always prefers being employed regardless of his action; so even if the worker can choose whether or not to resign, he would never do so. Thus, we do not explicitly model his resignation decision.

[10] In general, when $\tilde{a}$ is discontinuous and sometimes equal to 0, existence and uniqueness of p_t defined by (1) and (2) are not guaranteed. Although a complete characterization of $\bar{\mathcal{A}}$ is not present, we do know that it has the following properties. First, it contains the set of all Lipschitz functions (Itô, 1946). Second, it contains the set of all functions of bounded variation that are bounded away from 0 (Nakao, 1972). Third, it contains the set of all functions that are locally integrable and have no zero point (Engelbert and Schmidt, 1985). Fourth, if $\tilde{a} \in \bar{\mathcal{A}}$ and $\tilde{a}(p) = 0$, then p is an absorbing state, and the corresponding process p_t is not part of any equilibrium (see Claim 1 in the Appendix).

[11] That is, there exist $n \in \mathbb{N}$ and $0 = x_1 < x_2 < ... < x_n = 1$, such that for each $i \in \{1, ..., n-1\}$, there exists a Lipschitz function f_i on $[x_i, x_{i+1}]$ such that $f_i(p) = a(p)$ for all $p \in (x_i, x_{i+1})$. The assumption of no removable discontinuities is without loss, because if a is piecewise Lipschitz with some removable discontinuities, changing those finite removable discontinuities to continuous points does not affect any party's payoff.

2.3 Equilibrium Definition

Given a conjecture $\tilde{a}$ about type B worker's strategy and a Markov strategy profile (a, s), the type B worker's value at belief p is

$$v(p|a, s, \tilde{a}) = \mathbb{E}_{a,s}\left[\int_0^\infty (w + a)\, e^{-rt} e^{-\int_0^t s_\tau d\tau} dt\right], \tag{3}$$

where the expectation is taken over the (a_t, s_t) processes, which is driven by the underlying belief process p_t in (2).[12] Meanwhile, the firm's value is

$$\pi(p|s, \tilde{a}) = \mathbb{E}_{\tilde{a},s}\left[\int_0^\infty e^{-rt} e^{-\int_0^t s_\tau d\tau} d\tilde{\Pi}_t\right], \tag{4}$$

where $d\tilde{\Pi}_t = (y - \theta \tilde{a}_t)dt + dB_t$ is the law of motion for the profit process under the *conjectured* action process, and the expectation is taken over the $(\tilde{a}_t, s_t)$ processes driven by the underlying belief process p_t in (1).

We define Markov equilibria as follows.

Definition 2. *A **Markov equilibrium** consists of a Markov strategy profile (a^*, s^*) and a conjecture $\tilde{a}$, such that the following hold for all $p \in (0, 1)$:*

1. ***Correct conjecture:*** $\tilde{a} \equiv a^*$*;*
2. ***Firm optimality:*** $s^* \in \operatorname{argmax}_{s \in \mathcal{S}} \pi(p|s, \tilde{a})$*;*
3. ***Worker optimality:*** $a^* \in \operatorname{argmax}_{a \in \mathcal{A}} v(p|a, s^*, \tilde{a})$*;*
4. ***Instantaneous sequential optimality:*** *If $s^* = \infty$ is a neighborhood of p, we have $a^*(p) = \lim_{\Delta \downarrow 0} a_\Delta(p|a^*, s^*, a^*)$ where, for any $\Delta > 0$,*

$$a_\Delta(p|a, s, \tilde{a}) \equiv \operatorname{argmax}_{\hat{a} \in [0,1]} \left\{ (w + \hat{a})\Delta + e^{-r\Delta}\mathbb{E}\left[v(p_\Delta|a, s, \tilde{a})\middle| p_0 = p, a_t = \hat{a}, \tilde{a}_t = \tilde{a}_0 \forall t \in [0, \Delta)\right]\right\}$$

In the above definition, the first three requirements are standard. The last requirement, first introduced by Kuvalekar and Lipnowski (2018), is a refinement that regulates the off-equilibrium behavior of the worker. Specifically, suppose that the relationship is not terminated at t, but according to the firm's equilibrium strategy it should fire the worker for sure at all beliefs around the current one. In a continuous-time model, the worker's action does not affect his payoff, because he expects himself to be fired at any $t' > t$. The worker's indifference at such information sets could generate a plethora of off-equilibrium behavior, which complicates our analysis. To circumvent

[12] Note that the conjecture $\tilde{a}$ is used to characterize the law of motion (1) of p_t.

this problem, we impose "instantaneous sequential optimality" for equilibrium refinement: when deciding what action to take at reputation p_t where the relationship should have been terminated, the worker assumes that the relationship will still last for some time period $\Delta > 0$, and then come back to whatever equilibrium strategies induce. Sending Δ to 0, the limit action is what we assume the worker would take. For small Δ, the belief $p_{t+\Delta}$ is very close to p_t with high probability, so that the relationship will be terminated after Δ with probability close to 1, *regardless of the worker's action*. In this case, type B worker should just shirk as much as possible. As a result, under instantaneous sequential optimality, type B worker's off-equilibrium action is always equal to 1. Intuitively, once the worker finds out that the firm mistakenly employs him and expects it to be corrected soon, he tries to get the most out of the current relationship by not working at all.

3 Main Results

In our model, the firm always (weakly) prefers to employ the type G worker. Thus, as the worker's reputation worsens, the firm becomes more inclined to fire the worker. Such incentives of the firm indicate that, for type B worker, he feels safer to shirk when his reputation is good, while he wants to work harder in order to slow down belief worsening when his reputation is bad.

The following theorem, which characterizes the unique Markov equilibrium of this game for arbitrary parameters, confirms the above intuition. The equilibrium strategies are illustrated in Figure 1.

Theorem 1. *For any $y \in (0, 1)$ and $w > 0$, the unique equilibrium of this game is characterized as follows. There exist $0 < \underline{p} \leq \bar{p} < 1$ such that:*

1. *If $p < \underline{p}$, the firm never fires the worker, and the type B worker shirks to the maximum level when p is small enough while he starts working harder as p increases (i.e., as his reputation worsens).*

2. *If $p \in (\underline{p}, \bar{p})$, the firm randomizes between firing and keeping the worker with an increasing firing rate, and the type B worker works harder as p increases;*

3. *If $p > \bar{p}$, the firm fires the worker.*

While the proof details are relegated to the appendix, here we explain the key steps in finding the unique equilibrium. First, we prove a result which formalizes our intuition that in any equilibrium, the firm will keep (fire) the worker if his reputation is sufficiently good (bad), while randomizing if his reputation is intermediate. Second, whenever the relationship continues with

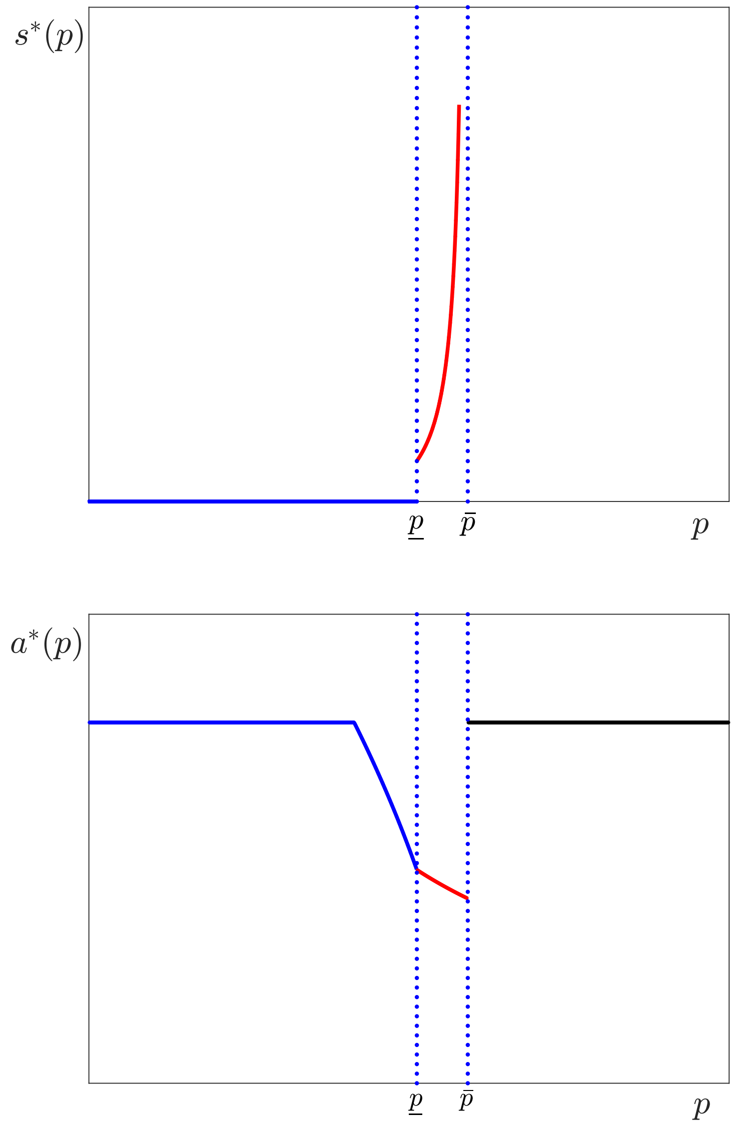

Figure 1: Equilibrium Strategies

positive probability, we can find closed-form solutions to type B worker's value and policy functions. Finally, optimality of the firm's decision (in particular, smooth pasting of its value function) allows us to uniquely pin down those belief cutoffs $\underline{p}$ and $\bar{p}$.

The equilibrium contains a region of beliefs within which the firm fires the worker stochastically. The following proposition, as illustrated in Figure 2, shows when such a region is nondegenerate.

Proposition 1. *For any $y \in (0,1)$, there exists $\bar{w}(y) \geq 0$ s.t. for all $w > \bar{w}(y)$, the equilibrium involves stochastic termination for a nondegenerate range of beliefs; that is, $\underline{p} < \bar{p}$. Moreover, $\bar{w}(y)$ is strictly decreasing in y.*

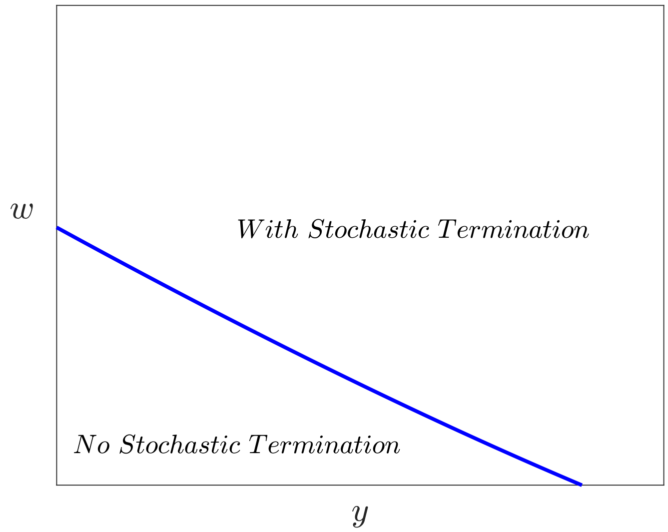

Figure 2: Range of Parameters Inducing Stochastic Termination

To understand this result, note first that type B worker's value is $\frac{1+w}{r}$ when $p = 0$, because 0 is an absorbing state at which the worker is never fired; note also that this value is increasing in w. In equilibrium, the firm's value falls from y/r to 0 as its belief moves from 0 to $\underline{p}$; meanwhile, type B worker's value falls from $\frac{1+w}{r}$ to some V. If w is very large, V would still be positive when $p = \underline{p}$; in order to deliver this positive value to the worker, the firm cannot terminate the relationship right away, but has to mix between its firing decisions with an increasing firing rate so as to gradually reduce the worker's value to 0.

Average Productivity and Job Insecurity

In this model, the worker's type is his private information which is never known to the firm/observer. From the firm/observer's perspective, the (average) productivity of the worker at belief p is given by $y - a^*(p)p$. Further, in the unique equilibrium, the worker's job becomes more and more insecure as p increases. The following proposition shows that the effect of job insecurity on productivity is ambiguous (in particular, U-shaped), which is consistent with findings in the organizational psychology literature (Selenko et al., 2013).

Proposition 2. *For any fixed* (w, y, r), *let* (a^*, s^*) *be the unique Markov equilibrium of the game. Then,* $y - a^*(p)p$ *first decreases and then increases with* p *on* $(0, \bar{p})$.

At each instant, if type B worker shirks more, he can get a higher instantaneous payoff; meanwhile, from the worker's viewpoint, holding fixed the firm's conjecture of his action, shirking more

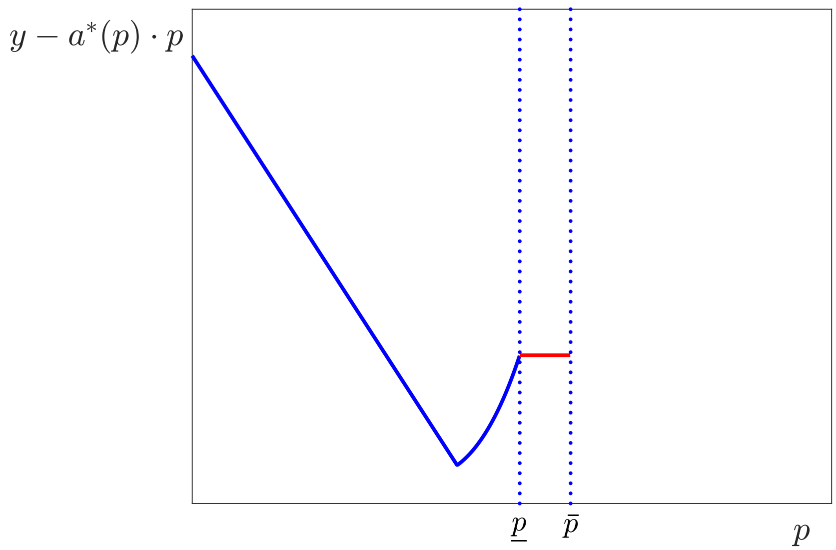

Figure 3: Average Productivity and Job Insecurity

reveals his bad type more quickly, and thus increases his probability of being fired. When deciding his action, type B worker's must balance these two forces.

When the worker's reputation is good (i.e., when p is small), marginally increasing the firm's belief will not significantly affect the firing probability as the threshold is still far away, so the worker's incentive to shirk is very high; indeed, he exerts minimal effort when p is small enough. Within this region, *average* productivity decreases with firm's belief as it becomes more probable that the worker exerts minimal effort.

As his reputation worsens and gets closer to the mixing region, the worker's job becomes more insecure, and type B worker's incentive to work hard gets stronger because he increasingly wants to slow down the firm's learning of his type. As a result, he exerts more effort while still working less hard than type G worker's default level. It turns out that the effort improvement from type B worker eventually dominates reputation worsening, so that *average* productivity starts to grow with the firm's belief.

When the worker's reputation reaches the mixing region, the firm's indifference requires that average productivity stays at 0, which is weakly increasing. Therefore, no matter whether the mixing region exists, the effect of job insecurity on productivity is U-shaped.

Comparison with Kuvalekar and Lipnowski (2018)

Our result regarding the relationship between productivity and job insecurity resembles that in Kuvalekar and Lipnowski (2018), who study a similar model under the assumptions of *costless* *and observable* effort and *symmetric* uncertainty about match quality between the firm and the worker. Though they (sometimes) also obtain a U-shape relationship between productivity and job insecurity, these results are different in the following ways.

First, the U-shape feature shows up in Kuvalekar and Lipnowski (2018) only when the equilibrium involves some stochastic termination; indeed, in their model, when the unique Markov equilibrium is in pure strategies, the worker's average productivity strictly decreases as his job becomes more insecure. In contrast, the U-shape relationship holds in our model even when the equilibrium only involves deterministic termination.[13]

Second, the economic reasons generating the equilibrium behavior are different. In our model with unobservable actions and asymmetric uncertainty, the worker's action affects the firm's learning through the *drift* term in (2), holding fixed the firm's conjecture and thus the diffusion term. In particular, a higher effort (i.e., a smaller a_t) from type B worker reduces the drift term and slows down the firm's learning of his type, because the induced output distribution gets closer to that generated by type G. When his job is fairly secure, type B worker exerts minimum effort to achieve the highest instantaneous payoff by saving the cost and as a result induces *fastest* learning (i.e., largest drift term in (2)); as his reputation worsens and unemployment becomes imminent, type B worker feels more urgent to slow down the firm's learning, and thus he increases his effort level, making his action better mimic type G's default action.

On the contrary, in their model with observable action and symmetric uncertainty, from *both* players' viewpoints the drift in the belief process always has zero expectation (i.e., belief is martingale), and learning is affected by the worker's action through the *variance/diffusion* term. In particular, a higher effort in their model increases belief variation. When his job is secure, in order to maintain its safe status, the worker exerts minimum effort to generate as little new information as possible. As the firm becomes more suspicious about match quality, the worker exerts higher effort to increase the *variance* of learning, hoping that a more positive signal can be realized to improve his reputation.

To summarize, the worker's incentive to achieve the highest instantaneous payoff results in the initial decreasing region of productivity in our model, while his incentive to generate least information leads to the same region in their model; furthermore, the worker's incentive to slow down learning generates the increasing region of productivity in our model, while his incentive to boost belief variance induces the counterpart of their model.

[13] However, if the wage w is too low, then it is possible in our model that the type B worker always exerts least effort until he is fired, in which case average productivity simply decreases with job insecurity.

4 Extensions

4.1 Costly Replacement

Our framework can be extended to a model with costly replacement. Suppose that there is a pool of workers and the proportion of them being the bad type is p_0. At time 0, the firm is randomly assigned a worker; at any instant, the firm can choose one from the following three actions: (i) keeping the worker; (ii) terminating the relationship and getting outside option zero; (iii) replacing the current worker with a new one at a cost of c. We consider stationary Markov equilibria where the firm and the worker always use the same Markov strategy regardless of how many workers the firm has hired before The stationary Markov equilibrium shares similar structure to that in the model without replacement. Indeed, when the replacement cost is high enough, in equilibrium the firm never replaces the worker, and their behavior is the same as in the baseline model. When the replacement is not so costly, instead of getting the outside option, in equilibrium the firm will replace the worker if his reputation is too bad, while the worker's behavior is still similar to that in the baseline model. The following proposition summarizes these results.

Proposition 3. *Given (w, y, r, p_0), there exists $\bar{c} \geq 0$, such that*

1. *If $c > \bar{c}$, the unique equilibrium is characterized by Theorem 1 under parameter values (w, y, r), in which the firm either keeps the worker or takes the outside option, but never replace the worker.*

2. *If $c < \bar{c}$, there is an equilibrium in which the firm either keeps the worker or replaces him, but never takes the outside option. Moreover, such an equilibrium has the same structure as that in Theorem 1 under parameter values (w, y_R, r) for some $y_R < y$.*

If replacement is not too costly, the firm has a positive continuation value after firing the current worker; in our model, this is equivalent to reducing the firm's instantaneous payoff from an employment relationship (from y to some y^r), while normalizing its continuation value after firing to 0.

4.2 Flexible Wage

Throughout our analysis, we have assumed that the wage is constant. As a robustness check, we show that our equilibrium characterization holds for a larger set of Markovian wage contracts with downward rigidity (possibly due to some minimum wage requirement).

Consider the following class $\mathcal{W}$ of Markovian wage contracts. Any $w \in \mathcal{W}$ is a bounded, weakly decreasing and continuously differentiable function of the firm's belief with bounded derivative; in addition, there is some $p_m \in (0,1)$ and $\underline{w} > 0$ such that $w(p) = \underline{w}$ for all $p \geq p_m$. Conditional on employment, any of such contracts has a minimum wage $\underline{\omega}$ that the employer cannot go under even if the worker's reputation is bad.

Proposition 4. *There exist $p^* \in (0,1)$ and $M > 0$, such that for any $w \in \mathcal{W}$ with $p_m < p^*$ and $\sup_p |w'(p)| < M$, any Markov equilibrium has the same structure as characterized in Theorem 1, and the average productivity is U-shaped as in Proposition 2.*

As a technical remark, the assumption that w is decreasing is not required; the above proposition holds as is if the wage contract w has bounded variation.

5 Conclusion

We develop a continuous-time model of employment relationship in which the worker faces moral hazard and reputation concerns, and the firm can only decide whether to retain or dismiss the worker without committing to a contract. We characterize the unique Markov equilibrium of this game. Our results suggest that the worker's moral hazard problem is mitigated as his reputation worsens, and that the relationship between average productivity and job insecurity is U-shaped. Besides, a stochastic termination region endogenously arises when the relationship is valuable enough. Our qualitative predictions still hold even when the firm can replace its current worker at a cost, or the wage varies subject to downward rigidity. In contrast to the contract theory literature, our paper provides an alternative angle for understanding how a firm's non-contractible firing threat can regulate its employee's incentive to shirk, when the latter has reputation concerns.

References

Atakan, Alp E., and Mehmet Ekmekci. 2012. "Reputation in Long-Run Relationships." *The Review of Economic Studies*, 79(2): 451–480.

Atakan, Alp E., and Mehmet Ekmekci. 2013. "A Two-Sided Reputation Result with Long-Run Players." *Journal of Economic Theory*, 148(1): 376 – 392.

Bhaskar, V., and George J. Mailath. 2019. "The Curse of Long Horizons." *Journal of Mathematical Economics*, 82: 74 – 89.

Bolton, Patrick, and Christopher Harris. 1999. "Strategic Experimentation." *Econometrica*, 67(2): 349–374.

Cisternas, Gonzalo. 2017. "Two-Sided Learning and the Ratchet Principle." *The Review of Economic Studies*, 85(1): 307–351.

Edmans, Alex, Xavier Gabaix, Tomasz Sadzik, and Yuliy Sannikov. 2012. "Dynamic CEO Compensation." *The Journal of Finance*, LXVII(5): 1603–1647.

Engelbert, H. J., and W. Schmidt. 1985. "On One-Dimensional Stochastic Differential Equations with Generalized Drift." 143–155. Springer Berlin Heidelberg.

Faingold, Eduardo, and Yuliy Sannikov. 2011. "Reputation in Continuous-Time Games." *Econometrica*, 79(3): 773–876.

Fudenberg, Drew, and David K. Levine. 1989. "Reputation and Equilibrium Selection in Games with a Patient Player." *Econometrica*, 57(4): 759–778.

Fudenberg, Drew, and David K. Levine. 1992. "Maintaining a Reputation when Strategies are Imperfectly Observed." *The Review of Economic Studies*, 59(3): 561–579.

Halac, Marina, and Andrea Prat. 2016. "Managerial Attention and Worker Performance." *American Economic Review*, 106(10): 3104–32.

Holmström, Bengt. 1999. "Managerial Incentive Problems: A Dynamic Perspective." *The Review of Economic Studies*, 66(1): 169–182.

Holmstrröm, Bengt. 1979. "Moral Hazard and Observability." *The Bell Journal of Economics*, 10(1): 74.

Itô, Kiyosi. 1946. "On a Stochastic Integral Equation." *Proc. Japan Acad.*, 22(2): 32–35.

Kreps, David M, and Robert Wilson. 1982. "Reputation and Imperfect Information." *Journal of Economic Theory*, 27(2): 253 – 279.

Kuvalekar, Aditya, and Elliot Lipnowski. 2018. "Job Insecurity." Working paper.

Mailath, G. J., and L. Samuelson. 2001. "Who Wants a Good Reputation?" *The Review of Economic Studies*, 68(2): 415–441.

Milgrom, Paul, and John Roberts. 1982. "Predation, Reputation, and Entry Deterrence." *Journal of Economic Theory*, 27(2): 280 – 312.

Nakao, Shintaro. 1972. "On the Pathwise Uniqueness of Solutions of One-Dimensional Stochastic Differential Equations." *Osaka J. Math.*, 9(3): 513–518.

Probst, Tahira M., Susan M. Stewart, Melissa L. Gruys, and Bradley W. Tierney. 2007. "Productivity, Counterproductivity and Creativity: The Ups and Downs of Job Insecurity." *Journal of Occupational and Organizational Psychology*, 80(3): 479–497.

Reisel, William D., Tahira M. Probst, Swee-Lim Chia, Cesar M. Maloles, and Cornelius J. Knig. 2010. "The Effects of Job Insecurity on Job Satisfaction, Organizational Citizenship Behavior, Deviant Behavior, and Negative Emotions of Employees." *International Studies of Management & Organization*, 40(1): 74–91.

Roskies, Ethel, and Christiane Louis-Guerin. 1990. "Job Insecurity in Managers: Antecedents and Consequences." *Journal of Organizational Behavior*, 11(5): 345–359.

Sannikov, Yuliy. 2008. "A Continuous-Time Version of the Principal-Agent Problem." *Review of Economic Studies*, 75: 957–984.

Selenko, Eva, Anne Mkikangas, Saija Mauno, and Ulla Kinnunen. 2013. "How Does Job Insecurity Relate to Self-Reported Job Performance? Analysing Curvilinear Associations in a Longitudinal Sample." *Journal of Occupational and Organizational Psychology*, 86(4): 522–542.

Sverke, Magnus, and Johnny Hellgren. 2001. "Exit, Voice and Loyalty Reactions to Job Insecurity in Sweden: Do Unionized and Non-unionized Employees Differ?" *British Journal of Industrial Relations*, 39(2): 167–182.

A Proofs

The technical structure of our problem resembles that of Kuvalekar and Lipnowski (2018). Their paper proves a number of results on the technical properties of equilibrium strategies and value functions. To be able to apply their results, however, we need to establish that the worker's strategy a^* in any equilibrium is bounded away from 0 on its domain $(0, 1)$.

In this appendix, we proceed as follows. In A.1, we show that a^* in any equilibrium is indeed bounded away from 0 (Lemma 2), which validates the technical lemmas proved in Kuvalekar and Lipnowski (2018) (Lemmas 3 through 5). In A.2, we characterize the unique features of equilibrium strategies (Lemmas 6 through 8), and reduce the research for an equilibrium to finding a finite dimensional vector. In A.3, we pin down the unique such vector satisfying the equilibrium conditions. Finally, in A.4, we establish several properties of the unique Markov equilibrium, proving the results in Section 3.

A.1 Technical Preliminaries

Recall that $\mathcal{A}$ is the set of piecewise Lipschitz functions with no removable discontinuities. We note that

$$\mathcal{A} = \{a : (0, 1) \to [0, 1] | a \text{ is piecewise Lipschitz with finite number of jump discontinuities}\}, \tag{5}$$

because on any open interval over which a is Lipschitz, the right (left) limit of a at the left (right) boundary point exists. This implies that any $a \in \mathcal{A}$ has one-sided limits everywhere on $(0, 1)$. Our first goal is to show that for the purpose of finding an equilibrium, it is without loss to restrict attention to

$$\mathcal{A}^* = \{a \in \mathcal{A} | \exists \epsilon_a > 0, \text{ s.t. } a(p) > \epsilon_a, \forall p \in (0, 1)\}. \tag{6}$$

To establish some necessary conditions of an equilibrium, let us take an equilibrium (a^*, s^*) as given. We define $v^* \equiv v(\cdot | a^*, s^*, a^*)$ and $\pi^* \equiv \pi(\cdot | a^*, s^*)$.

Claim 1. *In any equilibrium (a^*, s^*), $a^*(p) > 0$ for all $p \in (0, 1)$.*

Proof. Suppose (by contradiction) that there exists $\tilde{p} \in (0, 1)$ s.t. $a^*(\tilde{p}) = 0$. Since, in equilibrium, the firm's conjecture satisfies $\tilde{a} = a^*$, the law of motion of the belief process in (1) implies that $\tilde{p}$ is an absorbing state. It cannot be that $s^*(\tilde{p}) = \infty$, for otherwise if the belief process reaches $\tilde{p}$, by deviating to $s \equiv 0$ the firm can get a payoff of $y > 0$ at belief $\tilde{p}$, instead of 0. So we must have $s^*(\tilde{p}) < \infty$, which implies that the (type B) worker's value at the absorbing state $\tilde{p}$ is $\frac{ra^*(\tilde{p})}{r+s^*(\tilde{p})}$. Apparently, worker optimality requires that $a^*(\tilde{p}) = 1$, a contradiction to $a^*(\tilde{p}) = 0$. $\square$

Claim 1 indicates that, without loss of generality, the firm's conjecture can be restricted to the following space.

$$\mathcal{A}^0 = \{a \in \mathcal{A} | a(p) > 0, \forall p \in (0, 1)\}.$$

Lemma 1. *Fix $0 < p_0 < p_1 < 1$, worker's strategy $a \in \mathcal{A}$, stopping rate $s \in \mathcal{S}$ and conjecture $\tilde{a} \in \mathcal{A}^0$. Suppose that, on $[p_0, p_1]$, s is Lipschitz and finite, and $a, \tilde{a}$ are Lipschitz. Let $v \equiv v(\cdot|a, s, \tilde{a})$ and $v^* \equiv \sup_{a\in\mathcal{A}} v(\cdot|a, s, \tilde{a})$. Then v, v^* are C^2 on $[p_0, p_1]$ and satisfy*

$$(r + s(p))v(p) = w + a(p) + p(1-p)\left[a(p)\tilde{a}(p) - p\tilde{a}(p)^2\right] v'(p) + \frac{1}{2}p^2(1-p)^2\tilde{a}(p)^2 v''(p). \tag{7}$$

$$(r + s(p))v^*(p) = \sup_{\hat{a}\in[0,1]} w + \hat{a} + p(1-p)\left[\hat{a}\tilde{a}(p) - p\tilde{a}(p)^2\right] v^{*\prime}(p) + \frac{1}{2}p^2(1-p)^2\tilde{a}(p)^2 v^{*\prime\prime}(p). \tag{8}$$

Moreover, any boundary value problem of differential equation (7) admits at most one solution.

Proof. Since $\tilde{a}(p) > 0$ and is Lipschitz on $[p_0, p_1]$, by Weierstrass theorem $\min_{p\in[p_0,p_1]} \tilde{a}(p) > 0$, so that the belief process in (1) is uniformly nondegenerate on $[p_0, p_1]$. The result then follows from Theorem 5 in Chapter 1 of Krylov (2008). □

Claim 2. *In any equilibrium (a^*, s^*), if there exist $0 \le p_0 < p_1 \le 1$ s.t. $s^*|_{(p_0,p_1)} > 0$, then $y - a^*(p)p \le 0$ at all but finite $p \in (p_0, p_1)$; moreover, if $s^*|_{(p_0,p_1)} \in (0, \infty)$, then $y - a^*(p)p = 0$ at all but finite $p \in (p_0, p_1)$.*

Proof. Suppose that $s^*|_{(p_0,p_1)} > 0$. Then, the worker's indifference, if any, implies that $\pi^*(p) = 0$ for all $p \in (p_0, p_1)$. At any continuous point $\tilde{p} \in (p_0, p_1)$ of a^*, suppose (by contradiction) that $y - a^*(\tilde{p})\tilde{p} > 0$. Then there is a neighborhood $I \subsetneq (p_0, p_1)$ of $\tilde{p}$, s.t. $y - a^*(p)p > 0$ for all $p \in I$. Consider another stopping rule $\tilde{s}$ defined by

$$\tilde{s} = \begin{cases} 0, & \text{if } p \in I \\ \infty, & \text{if } p \notin I \end{cases}.$$

Obviously, $\pi(\tilde{p}|a^*, \tilde{s}) > 0 = \pi^*(\tilde{p})$, but this is a contradiction to the optimality of s^* at $\tilde{p}$. Since a^* only has finite discontinuities, $y - a^*(p)p \le 0$ for all but finite $p \in (p_0, p_1)$.

Now suppose that $s^*|_{(p_0,p_1)} \in (0, \infty)$. Suppose (by contradiction) that $y - a^*(\tilde{p})\tilde{p} < 0$. Then there is a neighborhood $I \subsetneq (p_0, p_1)$ of $\tilde{p}$, s.t. $y - a^*(p)p < 0$ for all $p \in I$. Let $\tau(\tilde{p}) = \inf\{t : p_t \notin I, p_0 = \tilde{p}\}$. The firm's indifference and optimality imply that

$$\begin{aligned} 0 = \pi^*(\tilde{p}) &= \mathbb{E}_\tau\left[\int_0^{\tau(\tilde{p})} e^{-rt}(y - a^*(p_t)p_t)dt + e^{-r\tau(\tilde{p})}\pi^*(p_{\tau(\tilde{p})})\right] \\ &= \mathbb{E}_\tau\left[\int_0^{\tau(\tilde{p})} e^{-rt}(y - a^*(p_t)p_t)dt\right] \\ &< 0 \end{aligned}$$

where the second line follows from $\pi^*(p_{\tau(\tilde{p})}) = 0$ everywhere because $I \subsetneq (p_0, p_1)$. But $0 < 0$ is a contradiction. □

Claim 3. *In any equilibrium (a^*, s^*), fixing $0 \le p_0 < p_1 \le 1$ such that $a^*|_{(p_0,p_1)}$ is continuous and $s^*|_{(p_0,p_1)}$ is continuous and finite, then on (p_0, p_1),*

$$a^*(p) = \frac{1}{\max\{1, -v^{*\prime}(p)p(1-p)\}}. \tag{9}$$

Further,

- *if $s^*|_{(p_0,p_1)} = 0$ and $a^*|_{(p_0,p_1)} < 1$, then*

$$v^*(p) = \frac{w}{r} + \frac{1}{\sqrt{2r}}\Phi^{-1}\left(A_1 \frac{p}{1-p} + A_2\right), \tag{10}$$

for some $A_1 < 0$ and $A_2 \in \mathbb{R}$, where Φ is the CDF of standard normal distribution.

- *if $s^*|_{(p_0,p_1)} = 0$ and $a^*|_{(p_0,p_1)} = 1$, then*

$$v^*(p) = \frac{1+w}{r} + B_1\left(\frac{p}{1-p}\right)^{\beta_1} + B_2\left(\frac{p}{1-p}\right)^{\beta_2}, \tag{11}$$

for some $B_1, B_2 \in \mathbb{R}$, where β_1, β_2 are roots of $\frac{1}{2}x(x+1) = r$, satisfying $\beta_1 > 0 > \beta_2$.

- *if $s^*|_{(p_0,p_1)} \in (0, \infty)$, then*

$$a^*(p) = \frac{y}{p}, \tag{12}$$

$$v^*(p) = \frac{\ln(1-p)}{y} + C, \tag{13}$$

$$s^*(p) = \frac{w + \frac{y}{2}}{v^*(p)} - r, \tag{14}$$

for some $C \in \mathbb{R}$.

Proof. Suppose that $a^*|_{(p_0,p_1)}$ is continuous and $s^*|_{(p_0,p_1)}$ is continuous and finite. Then on any closed interval $I \subset (p_0, p_1)$, $a^*|_I$ is Lipschitz and $s^*|_I$ is Lipschitz and finite. So Lemma 1 applies to all such I's.

Fix any closed interval $I \subset (p_0, p_1)$. By Lemma 1, we know that (8) is satisfied on I. Note that the RHS of (8) is affine in its choice variable $\hat{a}$, with coefficient $1 + p(1-p)\tilde{a}(p)v^{*\prime}(p)$. Optimality of a^* requires that, almost everywhere on I,

$$a^*(p) \begin{cases} = 1, & \text{if } 1 + p(1-p)\tilde{a}(p)v^{*\prime}(p) > 0 \\ \in [0,1], & \text{if } 1 + p(1-p)\tilde{a}(p)v^{*\prime}(p) = 0 \\ = 0, & \text{if } 1 + p(1-p)\tilde{a}(p)v^{*\prime}(p) < 0 \end{cases}. \tag{15}$$

In equilibrium $\tilde{a} = a^*$, so condition (15) implies that (9) must hold almost everywhere on I. Since a^* and $\nu^{*\prime}$ are continuous on I, (9) must hold everywhere on I. Applying the above argument to $I_n \equiv \left[p_0 + \frac{1}{n}, p_1 - \frac{1}{n}\right]$, we conclude that (9) holds on (p_0, p_1).

Now, let us first suppose that $s^*|_{(p_0,p_1)} = 0$ and $a^*|_{(p_0,p_1)} < 1$. By condition (9), we must have $a^*(p) = -\frac{1}{v^{*\prime}(p)p(1-p)} > 0$ on (p_0, p_1). This implies that $v^{*\prime}|_{(p_0,p_1)} < 0$. Substituting it into (8), we get

$$rv^*(p) = w - \frac{1}{v^{*\prime}(p)(1-p)} + \frac{1}{2}\frac{v^{*\prime\prime}(p)}{[v^{*\prime}(p)]^2}.$$

The general solution to the above second-order differential equation is given by (10).

Next, suppose that $s^*|_{(p_0,p_1)} = 0$ and $a^*|_{(p_0,p_1)} = 1$. Substituting it into (8), we get

$$rv^*(p) = w + 1 + p(1-p)^2 v^{*\prime}(p) + \frac{1}{2}p^2(1-p)v^{*\prime\prime}(p).$$

The general solution to the above second-order differential equation is given by (11).

Finally, suppose that $s^*|_{(p_0,p_1)} \in (0, \infty)$. By Claim 2, $a^*|_{(p_0,p_1)}$ must satisfy that $a^*(p)p = y$, i.e., $a^*(p) = y/p$ as in (12). By condition (9), we have

$$y/p = -\frac{1}{v^{*\prime}(p)p(1-p)},$$

which implies that $v^*|_{(p_0,p_1)}$ satisfies (13). Finally, Lemma 1 implies that s^* must be such that (14) holds almost everywhere on (p_0, p_1). As $s^*|_{(p_0,p_1)}$ is continuous, (14) must hold everywhere on (p_0, p_1). □

Lemma 2. *In any equilibrium* (a^*, s^*), $a^* \in \mathcal{A}^*$ *where* $\mathcal{A}^*$ *is defined in* (6).

Proof. Since a^* is piecewise Lipschitz, its one-sided limit exists everywhere on $(0, 1)$. Since $a^*(p) \neq 0$ for all $p \in (0, 1)$, we are done if we can show that at any $\tilde{p}$ of the finite discontinuities of a^*, $a^*_-(\tilde{p}), a^*_+(\tilde{p}) > 0$.

We first look at its left limit $a^*_-(\tilde{p})$. Since a^* is piecewise continuous, there exists $\epsilon > 0$ s.t. a^* is continuous on $(\tilde{p} - \epsilon, \tilde{p})$. Suppose (by contradiction) that $a^*_-(\tilde{p}) = 0$. Then there exists $\epsilon_1 \leq \epsilon$ s.t. $y - a^*(p)p > 0$ for all $p \in (\tilde{p} - \epsilon_1, \tilde{p})$, and then firm optimality requires that $s^*(p) < \infty$ for all $p \in (\tilde{p} - \epsilon_1, \tilde{p})$. Piecewise continuity of s^* indicates that we can without loss assume s^* to be continuous on $(\tilde{p} - \epsilon_1, \tilde{p})$. Note that now both a^* and s^* satisfy the condition in Claim 3 on $(\tilde{p} - \epsilon_1, \tilde{p})$, so Claim 3 tells us that

$$a^*(p) = \frac{1}{\max\{1, -v^{*\prime}(p)p(1-p)\}},$$

which implies that $v^{*\prime}_-(\tilde{p}) = -\infty$. Now we argue that this is not possible.

To see this, Claim 3 indicates that if $s^*(\hat{p}) > 0$ for some $\hat{p} \in (\tilde{p} - \epsilon_1, \tilde{p})$, then $s^*(p) > 0$ for all $p \in (\hat{p}, p)$. This is because $s^*(p)$ defined by (14) in Claim 3 is strictly increasing in p, so that if $s^*(p)$ drops back to 0 within $(\hat{p}, p)$, then it violates continuity. This implies that there exists $\epsilon_2 \leq \epsilon_1$ s.t. either $s^*|_{(\tilde{p}-\epsilon_2,\tilde{p})} = 0$ or $s^*|_{(\tilde{p}-\epsilon_2,\tilde{p})} \in (0, \infty)$. As $y - a^*(p)p > 0$ for all $p \in (\tilde{p} - \epsilon_2, \tilde{p})$, Claim 2 tells us that the latter case is not possible. Meanwhile, if $s^*|_{(\tilde{p}-\epsilon_2,\tilde{p})} = 0$, then v^* is given by (10). Note that

$$v^{*\prime}(p) = \frac{A_1}{\sqrt{2r}} \frac{1}{(1-p)^2 \phi\left[\sqrt{2r}\,(v^*(p) - w/r)\right]},$$

where ϕ is the PDF of standard normal distribution. In this case, $v^{*\prime}(p)$ is bounded on $(\tilde{p}-\epsilon_2,\tilde{p})$ as $v^*(p)$ is bounded (by $(w+1)/r$). This is a contradiction to $v^{*\prime}_{-}(\tilde{p})=-\infty$.

An argument very similar to above shows that $a^*_{+}(\tilde{p})>0$, so we are done. □

Lemma 2 implies that for the purpose of finding an equilibrium, we can restrict firm's conjecture $\tilde{a}$ to be from $\mathcal{A}^*$, though the worker's strategy space, especially when considering deviations, remains to be the original $\mathcal{A}$. As defined in (6), any $\tilde{a}\in\mathcal{A}^*$ is such that it is bounded away from 0 by some number $\epsilon_{\tilde{a}}$ (which can be $\tilde{a}$-dependent). From (1), we know that for any fixed $\tilde{a}\in A^*$, the belief process p_t does not have any absorbing state (because the coefficient on dB_t is always positive) for all $a\in\mathcal{A}$. These enable us to draw from the results in Kuvalekar and Lipnowski (2018), and extend our analysis of equilibrium structure to the entire domain $(0,1)$.

Lemma 3. *Given any $a\in\mathcal{A}$, $s\in\mathcal{S}$ and $\tilde{a}\in\mathcal{A}^*$, the value functions $v(\cdot|a,s,\tilde{a})$ and $\pi(\cdot|\tilde{a},s)$ are continuous.*

Proof. This follows from the proof of Lemma 1 of Kuvalekar and Lipnowski (2018). □

Lemma 4. *Fix a conjecture $\tilde{a}\in\mathcal{A}^*$. Let $\pi^*\equiv\sup_{s\in\mathcal{S}}\pi(\cdot|\tilde{a},s)$, and suppose $s^*\in\operatorname{argmax}_{s\in\mathcal{S}}\pi(\cdot|\tilde{a},s)$. Fixing $0\le p_0<p_1\le 1$ s.t. $s^*|_{(p_0,p_1)}=0$, then on (p_0,p_1), π^* is piecewise C^2 and satisfies*[14]

$$r\pi^*(p)=y-\tilde{a}(p)p+\frac{1}{2}p^2(1-p)^2\tilde{a}(p)^2\pi^{*\prime\prime}(p); \tag{16}$$

Proof. On any subinterval over which $\tilde{a}$ is Lipschitz, (16) follows from Theorem 5 in Chapter 1 of Krylov (2008). The lemma follows because $\tilde{a}$ is piecewise Lipschitz. □

Lemma 5. *Take any $s\in\mathcal{S}$. Let $p\in(0,1)$ be such that $s(p)<\infty$. Given any $a\in\mathcal{A}$, $s\in\mathcal{S}$ and $\tilde{a}\in\mathcal{A}^*$, $v(\cdot|a,s,\tilde{a})$ is continuously differentiable at p.*

Proof. This follows from the proofs of Lemmas 4 and 5 of Kuvalekar and Lipnowski (2018). □

A.2 Establishing the Unique Equilibrium Structure

The following three lemmas give a characterization of the equilibrium structure. We show that, in any equilibrium, the firm continues the relationship when p is low, it stochastically terminates the relationship when p is intermediate, and it fires the worker for sure when p is high (Lemma 7). Moreover, we provide closed-form formula to type B worker's equilibrium strategy (Lemmas 6 and 8).

[14] Except at the finite number of discontinuities of $\tilde{a}$ on (p_0,p_1).

Define $h : (0, 1) \to (0, \infty)$ by

$$h(p) \equiv \frac{p}{1-p}. \tag{17}$$

For notational convenience, we sometimes write players' value and policy functions using h as the state variable, which is equivalent to using p as the state variable because h is a strictly increasing transformation of p. In particular, given an equilibrium (a^*, s^*) and its associated value functions v^* and π^*, we can define $\hat{a}, \hat{s}, \hat{v}$ and $\hat{\pi}$ by

$$\begin{aligned} \hat{a}(h) &= a^*\left(\frac{h}{1+h}\right), \\ \hat{s}(h) &= s^*\left(\frac{h}{1+h}\right), \\ \hat{v}(h) &= v^*\left(\frac{h}{1+h}\right), \\ \hat{\pi}(h) &= \pi^*\left(\frac{h}{1+h}\right). \end{aligned} \tag{18}$$

The following lemma, which will be repeatedly used, generalizes Claim 3.

Lemma 6. *In any equilibrium (a^*, s^*), fixing $0 \le p_0 < p_1 \le 1$, if $s^*|_{(p_0,p_1)} < \infty$, then $a^*|_{(p_0,p_1)}$ is continuous and satisfies (9). Furthermore,*

1. *if $s^*|_{(p_0,p_1)} = 0$, then on any open interval $I \subset (p_0, p_1)$,*
   - *if $a^*|_I < 1$, then $v^*|_I$ satisfies (10), and $a^*(p)$ on I must be increasing, or decreasing, or first increasing and then decreasing; same for $p \cdot a^*(p)$.*
   - *if $a^*|_I = 1$, then $v^*|_I$ satisfies (11).*
2. *if $s^*|_{(p_0,p_1)} \in (0, \infty)$, then on (p_0, p_1), a^*, v^* and s^* satisfy (12), (13) and (14), respectively; a^* and v^* are strictly decreasing, and s^* is strictly increasing.*
3. *if $s^*|_{(p_0,p_1)} = \infty$, then on (p_0, p_1), $a^* = 1, v^* = 0$.*

Proof. Compared to Claim 3, Lemma 6 removes the restriction that a^* and s^* are continuous on (p_0, p_1). So Claim 3 implies that (9) holds at all $p \in (p_0, p_1)$ but finite discontinuities of a^* and s^*. By Lemma 5, the RHS of (9) is continuous everywhere on (p_0, p_1). Recalling condition (5), all of the finite number of discontinuous points of a^* are jump discontinuities; thus, that (9) holds a.e. on (p_0, p_1) implies that (9) holds on the entire interval (p_0, p_1).[15]

[15] Note first that for any piecewise Lipschitz function, one-sided limits are well defined everywhere. Take any $a \in \mathcal{A}$ s.t. $a(\tilde{p}) \neq \frac{1}{\max\{1, -v^{*\prime}(\tilde{p})\tilde{p}(1-\tilde{p})\}}$ for some $\tilde{p} \in (p_0, p_1)$. Let us show that a violates (9) on some interval. If a is continuous at $\tilde{p}$, obviously $a(p) \neq \frac{1}{\max\{1, -v^{*\prime}(p)p(1-p)\}}$ in a neighborhood of $\tilde{p}$. If a is discontinuous at $\tilde{p}$, since all discontinuous points of a are jump discontinuities, we have $\lim_{p_n \uparrow \tilde{p}} a(p_n) \neq \lim_{p_n \downarrow \tilde{p}} a(p_n)$. Without loss, let us assume $\lim_{p_n \uparrow \tilde{p}} a(p_n) \neq \frac{1}{\max\{1, -v^{*\prime}(\tilde{p})\tilde{p}(1-\tilde{p})\}}$. Since a is continuous on $(\tilde{p} - \epsilon, \tilde{p})$ for some $\epsilon > 0$, we have $a(p) \neq \frac{1}{\max\{1, -v^{*\prime}(p)p(1-p)\}}$ for all $p \in (\tilde{p} - \epsilon, \tilde{p})$. In both cases, we do not have (9) hold almost everywhere.

Now we move on to the numbered points in the lemma.

For Part 1, since we have shown that a^* is continuous on (p_0, p_1), Claim 3 applies so that we only need to establish the statement regarding monotonicity a^* on any open interval $I \subset (p_0, p_1)$ such that $a^*|_I < 1$.

Recall that $h \equiv \frac{p}{1-p}$. By (2) and Ito's Lemma, the law of motion of h_t (from type B worker's viewpoint) is

$$dh_t = \tilde{a}_t h_t \left(a_t dt - dB_t\right)$$

Recall the definitions of $\hat{v}$ and $\hat{a}$ in (18). By the chain rule, $v^{*\prime}(p) = \hat{v}'(h(p))\frac{1}{(1-p)^2}$. Then, by equation (9) and the assumption that $a^*|_I < 1$, we know that for all $h \in I_h \equiv (h(p_0), h(p_1))$,

$$\hat{a}(h) = -\frac{1}{\hat{v}'(h)h}. \tag{19}$$

By definition of h and $\hat{a}$, we have

$$\text{sgn}[a^{*\prime}(p)] = \text{sgn}[\hat{a}'(h(p))] = -\text{sgn}\left[\frac{d}{dh}\left(-\hat{v}'(h)h\right)\right]. \tag{20}$$

From (10), we can see that

$$\hat{v}(h) = \frac{w}{r} + \frac{1}{\sqrt{2r}}\Phi^{-1}(A_1 h + A_2), \tag{21}$$

thus $\hat{v}'(h) = \frac{A_1}{\sqrt{2r}\phi\left[\sqrt{2r}(\hat{v}(h)-w/r)\right]}$ where ϕ is the PDF of standard normal distribution. Equation (19) and that $\hat{a}|_{I_h} \in (0,1)$ then imply that $A_1 < 0$. Moreover,

$$\begin{aligned}
\frac{d}{dh}\left(-\hat{v}'(h)h\right) &= \frac{d}{dh}\frac{|A_1|h}{\sqrt{2r}\phi\left[\sqrt{2r}\left(\hat{v}(h) - w/r\right)\right]} \\
&= \frac{|A_1|}{\sqrt{2r}}\frac{\phi - h\sqrt{2r}\hat{v}'(h)\phi'}{\phi^2} \\
&= \frac{|A_1|}{\sqrt{2r}}\frac{\phi + h\hat{v}'(h)2r\left(\hat{v}(h) - w/r\right)\phi}{\phi^2} \\
&= \frac{|A_1|}{\phi\sqrt{2r}}\left[1 - \frac{2(r\hat{v}(h) - w)}{\hat{a}(h)}\right] \\
&= \frac{|A_1|}{\phi\sqrt{2r}}\left[1 - \frac{\sqrt{2r}\Phi^{-1}(A_1 h + A_2)}{\hat{a}(h)}\right]
\end{aligned} \tag{22}$$

where the third line follows from the property of the standard normal PDF that $\phi'(x) = -x\phi(x)$, the fourth line follows from (19), and the last line follows from (21). Note that $\frac{d}{dh}\left(-\hat{v}'(h)h\right)$ is continuous on I_h.

The above expression tells us that, if $\hat{a}'(h_2) > 0$ for some $h_2 \in I_h$, then $\hat{a}'(h) > 0$ for all $h \leq h_2$ on I_h. To see this, let us take such an h_2. Since $\hat{v}$ is C^2 (by Lemma 1) and $\hat{a}$ satisfies (19), $\hat{a}$ is C^1; hence, there exists $\epsilon > 0$ such that $\hat{a}'(h) > 0$ for all $h \in (h_2 - \epsilon, h_2]$. Then,

$$\begin{aligned}
\frac{d}{dh}\left(-\hat{v}'(h)h\right)\Big|_{(h_2-\epsilon,h_2)} &= \frac{|A_1|}{\phi\sqrt{2r}}\left[1 - \frac{\sqrt{2r}\Phi^{-1}(A_1 h + A_2)}{\hat{a}(h)}\right]\Bigg|_{(h_2-\epsilon,h_2)} \\
&< \frac{|A_1|}{\phi\sqrt{2r}}\left[1 - \frac{\sqrt{2r}\Phi^{-1}(A_1 h + A_2)}{\hat{a}(h)}\right]\Bigg|_{h=h_2} \\
&< \frac{d}{dh}\left(-\hat{v}'(h)h\right)\Big|_{h=h_2}
\end{aligned}$$

where the first inequality follows from $\hat{a}(h) < \hat{a}(h_2)$ and $\Phi^{-1}(A_1 h + A_2) > \Phi^{-1}(A_1 h_2 + A_2)$ on $(h_2 - \epsilon, h_2)$, recalling that $A_1 < 0$. Let

$$\tilde{h} = \inf\{k \in I_h : \hat{a}'(h) > 0, \forall h \in (k, h_2]\}.$$

We need to show that $\tilde{h} = \inf I_h$. Suppose (by contradiction) $\tilde{h} > \inf I_h$. Since $\hat{a}'(h)$ is continuous, it must be that $\hat{a}'(\tilde{h}) = 0$, which implies that $\frac{d}{dh}\left(-\hat{v}'(h)h\right)\big|_{h=\tilde{h}} = 0$. But since $\hat{a}'(h) > 0$ for all $h \in (k, h_2)$, by previous argument we know that $\frac{d}{dh}\left(-\hat{v}'(h)h\right)\big|_{(k,h_2]} < 0$ and is increasing at all h arbitrarily close to but greater than k, thus $\frac{d}{dh}\left(-\hat{v}'(h)h\right)$ cannot be continuous at $\tilde{h}$, a contradiction.

By (20), we conclude that $a^*(p)$ must be increasing, decreasing or first increasing and then decreasing on I. A completely analogous argument establishes the same property for $p \cdot a^*(p)$.

For Part 2, Claim 3 implies that (12), (13) and (14) hold at all $p \in (p_0, p_1)$ but finite discontinuities of a^* and s^*. Since a^* has no removable discontinuities, (12) must hold everywhere on (p_0, p_1). By Lemma 3, v^* is continuous, thus (13) holds everywhere on (p_0, p_1). This means that (14) holds at all $p \in (p_0, p_1)$ but finite discontinuities of s^*. We now show that s^* is continuous on (p_0, p_1). Suppose (by contradiction) that s^* is discontinuous at some $\tilde{p} \in (p_0, p_1)$. Since s^* has only finite discontinuities, there exists $\epsilon > 0$ s.t. s^* is continuous and (thus) satisfies (14) on $[\tilde{p} - \epsilon, \tilde{p})$ and $(\tilde{p}, \tilde{p} + \epsilon]$. Since v^* is continuous, from (14) we know that the left and right limits of s^* at $\tilde{p}$ are the same, a contradiction to s^* having no removable discontinuities.

For Part (3), if $s^*|_{(p_0,p_1)} = \infty$, apparently $v^*|_{(p_0,p_1)} = 0$. To show that $a^*|_{(p_0,p_1)} = 1$, we appeal to "instantaneous sequential rationality". Define $z_t = \log \frac{p_t}{1-p_t}$. From the law of motion of p_t in (2), one can easily verify that the law of motion of z_t (from type B worker's view point) is given by

$$dz_t = \tilde{a}_t \left[\left(a_t - \frac{1}{2}\tilde{a}_t\right) dt - dB_t\right]. \tag{23}$$

Suppose that $z \in (z_0, z_1)$ where $z_0 = \log \frac{p_0}{1-p_0}$ and $z_1 = \log \frac{p_1}{1-p_1}$. Given such z, define $\epsilon_0 \equiv z - z_0 > 0$ and $\epsilon_1 \equiv z_1 - z > 0$. We want to show that, for Δ small enough, $a_\Delta(p(z)|a^*, s^*, a^*) = 1$. To see this, recall that

$$a_\Delta(p|a, s, \tilde{a}) \equiv \text{argmax}_{\hat{a}\in[0,1]} \left\{ (w + \hat{a})\Delta + e^{-r\Delta}\mathbb{E}\left[v(p_\Delta|a, s, \tilde{a})|\, p_{t=0} = p, a_t = \hat{a}, \tilde{a}_t = \tilde{a}_0 \forall t \in [0, \Delta)\right] \right\}$$

We would like to argue that the marginal effect of changing $\hat{a}$ on the second term in the above maximization problem is negligible compared to that on the first term. Since $v^* = 0$ on (z_0, z_1), we have

$$\begin{aligned} & \left| \frac{d}{d\hat{a}} \mathbb{E}\left[v(p_\Delta | a, s^*, \tilde{a}) \right| p_{t=0} = p, a_t = \hat{a}, \tilde{a}_t = \tilde{a}_0 \forall t \in [0, \Delta)] \right| \\ \leq \quad & \frac{1+w}{r} \left(\left| \frac{d}{d\hat{a}} \Pr(z_\Delta - z < -\epsilon_0) \right| + \left| \frac{d}{d\hat{a}} \Pr(z_\Delta - z > \epsilon_1) \right| \right) \end{aligned}$$

For any $\hat{a}$,

$$z_\Delta - z \sim \mathcal{N}\left(\tilde{a}(z) \left(\hat{a} - \frac{1}{2}\tilde{a}(z) \right) \Delta, \tilde{a}(z)^2 \Delta \right)$$

Therefore,

$$\begin{aligned} \Pr(z_\Delta - z < -\epsilon_0) &= \Phi\left[-\frac{\epsilon_0}{\tilde{a}(z)^2 \Delta} - \frac{\hat{a} - \frac{1}{2}\tilde{a}(z)}{\tilde{a}(z)} \right] \\ \Pr(z_\Delta - z > \epsilon_1) &= \Phi\left[-\frac{\epsilon_1}{\tilde{a}(z)^2 \Delta} + \frac{\hat{a} - \frac{1}{2}\tilde{a}(z)}{\tilde{a}(z)} \right] \end{aligned}$$

As a result,

$$\left| \frac{d}{d\hat{a}} \Pr(z_\Delta - z < -\epsilon_0) \right|, \left| \frac{d}{d\hat{a}} \Pr(z_\Delta - z > \epsilon_1) \right| \sim \exp\left(-\frac{1}{\Delta^2} \right)$$

In contrast, the marginal effect of $\hat{a}$ on the first term in the maximization problem is Δ, which is of lower order than $\exp\left(-\frac{1}{\Delta^2}\right)$. So $a_\Delta(p(z)|a^*, s^*, a^*) = 1$ for Δ small enough, and thus $a^*(p(z)) = 1$. Since z is arbitrary on (z_0, z_1), we conclude that $a^*|_{(p_0, p_1)} = 1$. □

The following Lemma establishes the structure of the firm's strategy in any Markov equilibrium.

Lemma 7. *In any equilibrium* (a^*, s^*)*, there exist* $0 < \underline{p} \leq \bar{p} < 1$ *s.t.* $s^*|_{(0,\underline{p})} = 0$*,* $s^*|_{(\underline{p},\bar{p})} \in (0, \infty)$*, and* $s^*|_{(\bar{p},1)} = \infty$.

Proof. We prove this lemma in several steps.

Step 1. There exists $\underline{p} > 0$ s.t. $s^*(p) = 0$ for all $p \in (0, \underline{p})$.

To see this, notice that for $p < y$, we have $y - a \cdot p > 0$ for all $a \in [0, 1]$. So in any equilibrium, we must have $s^*(p) = 0$ for $p < y$, for otherwise if $s^*(\tilde{p}) > 0$ at some $\tilde{p} < y$ which implies that $\pi(\tilde{p}|a^*, s^*) = 0$, then there is an alternative stopping rate s defined by

$$s(p) = \begin{cases} 0, & \text{if } p < y \\ \infty, & \text{if } p \geq y \end{cases}$$

which generates strictly positive value, $\pi(\tilde{p}|a^*, s) > 0$, to the firm at belief $\tilde{p}$; this is a contradiction to the optimality of s^* at $\tilde{p}$. Define

$$\underline{p} \equiv \sup\{k : s^*(p) = 0, \forall p \in (0, k)\}. \tag{24}$$

Note that $\underline{p} \geq y > 0$, and $s^*(p) = 0$ for all $p \in (0, \underline{p})$.

Step 2. There does not exist any open interval $I > \underline{p}$ s.t. $s^*(p) = 0$ for all $p \in I$.

To see this, suppose (by contradiction) that such an I exists, and let $\tilde{p} \in I$. Without loss, $I = (p_0, p_1)$ where

$$p_0 \equiv \inf\{k : s^*(p) = 0, \forall p \in (k, \tilde{p}]\},$$
$$p_1 \equiv \sup\{k : s^*(p) = 0, \forall p \in [\tilde{p}, k)\}.$$

Note that $p_0 > \underline{p}$, for otherwise if $p_0 = \underline{p}$, then either $\underline{p}$ is a removable discontinuity of s^* or the definition of $\underline{p}$ is violated.

Now we show that there exists $\epsilon > 0$ s.t. $s^*(p) > 0$ for all $p \in (p_0 - \epsilon, p_0) \cup (p_1, p_1 + \epsilon)$. First, since s^* is piecewise continuous, there exist $\epsilon_1 > 0$ s.t. s^* is continuous on $(p_0 - \epsilon_1, p_0)$ and $(p_1, p_1 + \epsilon_1)$.[16] By definition of p_0, there exists $p_2 \in (p_0 - \epsilon_1, p_0)$ s.t. $s^*(p_2) > 0$.

- If $s^*(p_2) \in (0, \infty)$, then by Lemma 6(2), s^* is strictly increase around p_2. Since s^* is continuous on (p_2, p_0), $s^*|_{(p_2,p_0)}$ is either strictly increasing and finite, or strictly increasing to ∞ and then staying at ∞. So $s^*|_{(p_2,p_0)} > 0$.

- If $s^*(p_2) = \infty$, since s^* is continuous on (p_2, p_0), we must have $s^*|_{(p_2,p_0)} = \infty$, for otherwise the strict increasing property of s^* when $s^*(p) \in (0, \infty)$ (recall Lemma 6(2)) will lead to a contradiction to the continuity of $s^*|_{(p_2,p_0)}$. So $s^*|_{(p_2,p_0)} > 0$.

By definition of p_1, there exists $p_3 \in (p_1, p_1 + \epsilon_1)$ s.t. $s^*(p_3) > 0$. By the same argument as above, we have $s^*|_{(p_3,p_1+\epsilon_1)} > 0$. Moreover, it is easy to see that such a p_3 can be chosen arbitrarily close to p_1, thus $s^*|_{(p_1,p_1+\epsilon_1)} > 0$. Finally, define $\epsilon \equiv \min\{p_0 - p_2, \epsilon_1\}$. Then we have $s^*(p) > 0$ for all $p \in (p_0 - \epsilon, p_0) \cup (p_1, p_1 + \epsilon)$.

Given such an ϵ, optimality of s^* implies that $\pi^*(p) = 0$ for all $p \in (p_0 - \epsilon, p_0) \cup (p_1, p_1 + \epsilon)$, so that $\pi^{*\prime}_-(p_0) = \pi^{*\prime}_+(p_1) = 0$. Since s^* is an optimal stopping rule, smooth pasting of π^* implies that $\pi^{*\prime}(p_0) = \pi^{*\prime}(p_1) = 0$. But then, to ensure that $\pi^*(p) \geq 0$ for p close to but greater than p_0, we must have $\pi^{*\prime\prime}_+(p_0) \geq 0$ and $\pi^{*\prime\prime}_-(p_1) \geq 0$. Moreover, since $s^*|_{(p_0,p_1)} = 0$, Lemma 4 implies that (16) is satisfied on (p_0, p_1), so that

$$y - a^*(p_0)p_0 = -\frac{1}{2}p_0(1 - p_0)a^*(p_0)^2\pi^{*\prime\prime}_+(p_0) \leq 0.$$

By a similar argument, we have

$$y - a^*(p_1)p_1 = -\frac{1}{2}p_1(1 - p_1)a^*(p_1)^2\pi^{*\prime\prime}_+(p_1) \leq 0.$$

[16] Such an ϵ can be found regardless of whether s^* is continuous at p_0 and p_1.

By Lemma 6(1), on (p_0, p_1), $p \cdot a^*(p)$ is increasing, decreasing, or first increasing and then decreasing. In any of case these cases, we must have

$$y - a^*(p)p \leq 0, \forall p \in (p_0, p_1).$$

Then, (16) implies that $\pi^{*\prime\prime}(p) \geq 0$ for all $p \in (p_0, p_1)$. Since $\pi^*(p_0) = \pi^*(p_1) = \pi^{*\prime}(p_0) = \pi^{*\prime}(p_1) = 0$, this can be true only if $\pi^*(p) = 0$ and $a^*(p)p = y$ for all $p \in (p_0, p_1)$; that is, $\frac{d}{dp}[a^*(p)p] = 0$ on $p \in (p_0, p_1)$. Since $s^*|_{(p_0,p_1)} = 0$, from the proof of Lemma 6(1), this implies that $v^*(p)$ is constant on (p_0, p_1), which implies that $a^*(p) = \frac{1}{\max\{1, -v^{*\prime}(p)p(1-p)\}} = 1$ on (p_0, p_1). But this is a contradiction to $a^*(p)p = y$ for all $p \in (p_0, p_1)$. So there does not exist any open interval $I > \underline{p}$ s.t. $s^*|_I = 0$.

Step 3. If $s^*(\tilde{p}) = 0$ and $\tilde{p} > \underline{p}$, then s^* is discontinuous at $\tilde{p}$.

To see this, take such a $\tilde{p}$ and suppose (by contradiction) that s^* is continuous at $\tilde{p}$. Since s^* only has a finite number of (isolated) discontinuities, there exists $\epsilon > 0$ s.t. s^* is continuous on $(\tilde{p} - \epsilon, \tilde{p}]$. By Step 2, there exists $p_3 \in (\tilde{p} - \epsilon, \tilde{p})$ s.t. $s^*(p_3) > 0$; in fact, we can also have $s^*(p_3) < \infty$ because s^* is continuous on $(\tilde{p} - \epsilon, \tilde{p}]$ and $s^*(\tilde{p}) = 0$. By the same reasoning as in the second paragraph of Step 2, $s^*|_{(p_3,\tilde{p})}$ must be strictly increasing and finite, or strictly increasing to ∞ and then staying at ∞. In any of these cases, we reach a contradiction to the continuity of s^* at $\tilde{p}$ with $s^*(\tilde{p}) = 0$.

Step 4. If $s^*|_I < \infty$ on some open interval $I > \underline{p}$, then $s^*|_I \in (0, \infty)$.

To see this, since s^* is piecewise continuous, by Step 3 s^* can have at most finite zero points in I. In any subinterval of I over which $s^* \neq 0$, Lemma 6(2) applies. Since v^* defined in (13) has only one undetermined constant, continuity of v^* implies that such a constant does not change on I, so that the RHS of (14) has well-defined left and right limits equal to each other everywhere on I. Since s^* does not have removable discontinuities, this implies that $s^*|_I$ satisfies (14), which is continuous and strictly increasing. So $s^*|_I \in (0, \infty)$.

Step 5. There does not exist $\hat{p}$ such that $s^*(p) \in (0, \infty)$ for all $p \in (\hat{p}, 1)$.

To see this, suppose (by contradiction) such a $\hat{p}$ exists. Lemma 6(2) indicates that $\pi^*(p) = \frac{\ln(1-p)}{y} + C$ on $(\hat{p}, 1)$ for some constant C, which implies that π^* is unbounded when p tends to 1, a contradiction to $\pi^* \leq \frac{y}{r}$ in any equilibrium.

Step 6. If $s^*(\hat{p}) = \infty$, then $s^*(p) = \infty$ for all $p \geq \hat{p}$.

To see this, let $\hat{p}$ be such that $s^*(\hat{p}) = \infty$. We first show that $s^*(p) = \infty$ for all $p \geq \hat{p}$ at which s^* is continuous. Suppose (by contradiction) that for some $\tilde{p} > \hat{p}$, $s^*(\tilde{p}) < \infty$ and s^* is continuous at $\tilde{p}$. Since s^* has only finite discontinuities, s^* is continuous at some open interval I containing $\tilde{p}$. Without loss, $I = (p_4, p_5)$ where

$$p_4 \equiv \inf\left\{k : s^*(p) < \infty, \forall p \in (k, \tilde{p}]\right\},$$
$$p_5 \equiv \sup\left\{k : s^*(p) < \infty, \forall p \in [\tilde{p}, k)\right\}.$$

We make several observations about p_4 and p_5. First, $p_5 < 1$. This is because by Step 4 $s^*|_{(p_4,p_5)} \in (0, \infty)$, and then that $p_5 = 1$ would contradict Step 5. Second, $s^*(p_4) = s^*(p_5) = \infty$. This is because by definitions of p_4 and p_5, we can find an increasing (decreasing) sequence converging to p_4 (p_5) such that the value of s^* is ∞ along the sequence. Since $s^{*-1}(\infty)$ is closed, we conclude that $s^*(p_4) = s^*(p_5) = \infty$. This further implies that $v^*(p_4) = v^*(p_5) = 0$.

Recall that by Step 4, $s^*|_{(p_4,p_5)} \in (0, \infty)$. Lemma 6(2) tells us that v^* is strictly decreasing on (p_4, p_5), which is a contradiction to $v^*(p_4) = v^*(p_5) = 0$. So $s^*(p) = \infty$ for all $p \geq \hat{p}$ at which s^* is continuous. Finally, since the discontinuities of s^* are isolated, $s^{*-1}(\infty)$ being closed implies that $s^*(p) = \infty$ for all $p \geq \hat{p}$.

Step 7. There exists $\hat{p}$ s.t. $s^*(\hat{p}) = \infty$.

To see this, since s^* is piecewise continuous on $(0, 1)$, we let $I \equiv (\tilde{p}, 1)$ be the last subinterval of $(0, 1)$ over which s^* is continuous. By Step 3, s^* does not have zero point in I; by Step 5, it cannot be $0 < s^*(p) < \infty$ for all $p \in (\tilde{p}, 1)$. So there must exist $\hat{p} \in (\tilde{p}, 1)$ s.t. $s^*(\hat{p}) = \infty$.

Let us now complete the proof of the lemma. Recall that $\underline{p}$ is already defined in (24). Now we define

$$\bar{p} \equiv \inf\{k : s^*(p) = \infty, \forall p \geq k\}. \tag{25}$$

By Steps 6 and 7, $\bar{p}$ is well-defined and satisfies $\underline{p} \leq \bar{p} < 1$. By definitions of $\underline{p}$ and $\bar{p}$, we have $s^*|_{(0,\underline{p})} = 0$ and $s^*|_{(\bar{p},1)} = \infty$. Moreover, if $\underline{p} < \bar{p}$, then by definition of $\bar{p}$ and Step 6, $s^*(p) < \infty$ for all $p \in (\underline{p}, \bar{p})$; finally, by Step 4, we have $s^*|_{(\underline{p},\bar{p})} \in (0, \infty)$. □

Parts 2 and 3 of Lemma 6 characterizes type B worker's behavior when p is in $(\underline{p}, \bar{p})$ or $(\bar{p}, 1)$. The following lemma extends Part 1 of Lemma 6 and provides a closed-form formula for type B worker's behavior when $p \in (0, \underline{p})$.

Define

$$V_m = \frac{1+w}{r} - \frac{1}{\beta_1}, \tag{26}$$

where β_1 is the positive root of $r = \frac{1}{2}x(x+1)$.

Lemma 8. *In any equilibrium (a^*, s^*), let $V \equiv v^*(\underline{p})$.*

1. *If $V < V_m$, then there exists $p_1 \in (0, \underline{p})$ s.t.*

$$a^*(p) = \begin{cases} 1, & \text{if } p \leq p_1 \\ \frac{1}{-v^{*\prime}(p)p(1-p)}, & \text{if } p_1 < p \leq \underline{p} \end{cases}, \tag{27}$$

where v^* on $(0, p_1)$ and $(p_1, \underline{p})$ is given by (11) and (10), respectively, with the undetermined con-

stants satisfying

$$\begin{cases} A_1 & = \frac{1}{\underline{h}} \left\{ \Phi \left[\sqrt{2r} \left(V - \frac{w}{r} \right) \right] - \Phi \left[\sqrt{2r} \left(V_m - \frac{w}{r} \right) \right] - \sqrt{2r} \phi \left[\sqrt{2r} \left(V_m - \frac{w}{r} \right) \right] \right\} \\ A_2 & = \Phi \left[\sqrt{2r} \left(V_m - \frac{w}{r} \right) \right] + \sqrt{2r} \phi \left[\sqrt{2r} \left(V_m - \frac{w}{r} \right) \right] \\ B_1 & = - \frac{1}{\beta_1 h_1^{\beta_1}} \\ B_2 & = 0 \end{cases}, \tag{28}$$

where $\underline{h} \equiv \frac{\underline{p}}{1-\underline{p}}$, $h_1 \equiv \frac{p_1}{1-p_1}$ *and it satisfies*

$$h_1 = -\frac{\sqrt{2r}}{A_1} \phi \left[\sqrt{2r} \left(V_m - \frac{w}{r} \right) \right]. \tag{29}$$

Moreover, v^* *is decreasing on* $(0, \underline{p})$ *with* $v^*(p_1) = V_m$*;* a^* *is decreasing on* $(p_1, \underline{p})$.

2. *If* $V \geq V_m$*, then* $a^*(p) = 1$ *for all* $p \in (0, \underline{p})$*, and* v^* *on* $(0, \underline{p})$ *is given by* (11) *with*

$$\begin{cases} B_1 & = - \frac{\left(\frac{1+w}{r} - V \right)}{\underline{h}^{\beta_1}} \\ B_2 & = 0 \end{cases}. \tag{30}$$

Proof. Suppose first that $V < V_m$. Let us first show that there exists $p_1 \in (0, \underline{p})$ s.t.

$$a^*(p) \begin{cases} = 1, & \text{if } p \in (0, p_1) \\ < 1, & \text{if } p \in (p_1, \underline{p}) \end{cases}. \tag{31}$$

We proceed in several steps.

Step 1. We show that $a^*(p) < 1$ for some $p \in (0, \underline{p})$. Suppose (by contradiction) that $a^*(p) = 1$ for all $p \in (0, \underline{p})$ (i.e., $\hat{a}(h) = 1$ for all $h \in (0, \underline{h})$). In this case, type B worker's value function $\hat{v}$ is given by (11) on $(0, \underline{h})$. Boundedness of $\hat{v}$ requires that $B_2 = 0$. Moreover, condition (9) then implies that $B_1 \beta_1 h^{\beta_1 - 1} h \geq -1$ for all $h \in (0, \underline{h})$, so that

$$\hat{v}(h) = \frac{1+w}{r} + B_1 h_1^{\beta_1} \geq \frac{1+w}{r} - \frac{1}{\beta_1} = V_m, \forall h \in (0, \underline{h}).$$

This is a contradiction to our assumption that $\hat{v}(\underline{h}) = V < V_m$.

By Step 1, let us pick any $h' \in (0, h)$ s.t. $\hat{a}(h') < 1$. Define

$$h_1 \equiv \inf\{k : \hat{a}(h) < 1, \forall h \in (k, h')\}. \tag{32}$$

Step 2. We show that $h_1 > 0$. Suppose (by contradiction) that $h_1 = 0$. By the definition of h_1 in (32), we have $\hat{a}(h) < 1$ for $0 < h < h'$. By Claim 3, type B worker's value function $\hat{v}$ on $(0, h')$ satisfies $\Phi \left(\sqrt{2r} \left(\hat{v} - \frac{w}{r} \right) \right) = \tilde{A}_1 h + \tilde{A}_2$ for some $\left(\tilde{A}_1, \tilde{A}_2 \right)$. In addition, his equilibrium action is given by

$\hat{a}(h) = \frac{-1}{\hat{v}'(h)h} = \frac{-\sqrt{2r}\phi(\sqrt{2r}\hat{v})}{\tilde{A}_1 h}$. Since $\hat{v}(h)$ is bounded (so that the denominator is bounded away from 0), $\hat{a}(h)$ tends to infinity as $h \to 0$, which is impossible because $\hat{a}$ is bounded by 1.

Step 3. We show that $\hat{a}(h) < 1$ for all $h \in (h_1, \underline{h})$. Suppose (by contradiction) that there exists $h'' \in (h_1, \underline{h})$ s.t. $\hat{a}(h'') = 1$. Because $h_1 > 0$, continuity of $\hat{a}$ and condition (32) imply that $\hat{a}(h_1) = 1$. By Lemma 6(1), $\hat{a}$ is increasing, or decreasing, or first increasing and then decreasing on (h_1, h''). Since $\hat{a}(h_1) = \hat{a}(h'') = 1$ and $\hat{a}$ is bounded above by 1, it can only be the case that $\hat{a}(h) = 1$ for all $h \in (h_1, h'')$, but this is a contradiction to $\hat{a}(h') < 1$.

Step 4. We show that $\hat{a}(h) = 1$ for all $h \in (0, h_1)$. Were it not the case, we can find some point between $(0, h_1)$ s.t. $\hat{a} < 1$, and then the same construction in Steps 2 and 3 would render a contradiction.

Having established the structure of a^* on $(0, \underline{p})$, Lemma 6 then implies that v^* is given by (11) when $p \in (0, p_1)$ and by (10) when when $p \in (p_1, \underline{p})$. That is,

$$\hat{v}(h) = \begin{cases} \frac{1+w}{r} + B_1 h^{\beta_1} + B_2 h^{\beta_2}, & \text{if } h \le h_1 \\ \frac{w}{r} + \frac{1}{\sqrt{2r}} \Phi^{-1}\left(A_1 h + A_2\right), & \text{if } h \in (h_1, \underline{h}) \end{cases}$$

where β_1 and β_2 are the roots of $r = \frac{1}{2}x(x+1)$ s.t. $\beta_2 < 0 < \beta_1$.

Now we characterize those undetermined constants (A_1, A_2, B_1, B_2) and p_1 as functions of V and $\underline{p}$ (and primitive parameters). Note first that the boundedness of $\hat{v}$ implies that $B_2 = 0$. The value-matching conditions at h_1 and $\underline{h}$ are

$$\Phi\left[\sqrt{2r}\left(\frac{1+w}{r} + B_1 h_1^{\beta_1} - \frac{w}{r}\right)\right] = A_1 h_1 + A_2 \tag{33}$$

$$\Phi\left[\sqrt{2r}\left(V - \frac{w}{r}\right)\right] = A_1 \underline{h} + A_2 \tag{34}$$

Moreover, continuity of $\hat{a}$ at h_1 implies the following two smooth-pasting conditions:

$$1 = -\hat{v}'(h)h\big|_{h=(h_1)_-} = -B_1\beta_1 h_1^{\beta_1 - 1} h_1 \tag{35}$$

$$1 = -\hat{v}'(h)h\big|_{h=(h_1)_+} = -\frac{\sqrt{2r}\phi\left[\sqrt{2r}\left(\frac{1+w}{r} + B_1 h_1^{\beta_1} - \frac{w}{r}\right)\right]}{A_1 h_1} \tag{36}$$

Direct computation shows that the unique solution to the system of equations (33)-(36) is given by (28) and (29).

Finally, that $B_1 < 1$ and $A_1 < 1$ imply that v^* is decreasing on $(0, \underline{p})$; that a^* is decreasing on $(p_1, \underline{p})$ follows from $a^*(p_1) = 1$ and Lemma 6(1).

Suppose now that $V \ge V_m$. The previous argument implies that, in general, either $a^*(p) = 1$ for all $p \in (0, \underline{p})$ or (31) holds for some $p_1 \in (0, \underline{p})$ with $v^*(p_1) = V_m$; the latter is impossible when $V \ge V_m$, because v^* is strictly decreasing on $(p_1, \underline{p})$ (as implied by condition (9)). The solution to v^* follows from Lemma 6(1) and the boundary condition that $v^*(\underline{p}) = V$. □

Lemmas 6 through 8 indicate that the equilibrium strategies are pinned down by $(\underline{p}, \bar{p}, V)$, where $\underline{p}$ and $\bar{p}$ are the cutoffs in the firm's strategy, and V is type B worker's value at belief $\underline{p}$.

A.3 Solving for the Markov Equilibrium

We will provide necessary conditions to pin down a unique vector $(\underline{p}, \bar{p}, V)$, and then show that the corresponding candidate strategy profile (given by Lemmas 6 through 8) is a Markov equilibrium.

Given an equilibrium (a^*, s^*), let $\underline{p}$ be the lower cutoff in the firm's strategy. Recalling that $\underline{h} = \frac{\underline{p}}{1+\underline{p}}$, we define $\alpha : (0, \underline{h}) \to (0, 1)$ by

$$\alpha = \frac{h}{\underline{h}}. \tag{37}$$

For $p \in (0, \underline{p})$ (equivalently, for $h \in (0, \underline{h})$), we can without loss write their strategies as functions of α. Specifically, given $(\hat{a}, \hat{s}, \hat{v}, \hat{\pi})$ defined in (18), we define $\check{a}, \check{s}, \check{v}, \check{\pi} : (0, 1] \to \mathbb{R}$ by

$$\begin{aligned} \check{s}(\alpha; \underline{h}) &= \hat{s}\left(\alpha \underline{h}\right) \\ \check{\pi}(\alpha; \underline{h}) &= \hat{\pi}\left(\alpha \underline{h}\right) \\ \check{a}(\alpha; \underline{h}) &= \hat{a}\left(\alpha \underline{h}\right) \\ \check{v}(\alpha; \underline{h}) &= \hat{v}\left(\alpha \underline{h}\right) \end{aligned} \tag{38}$$

To be sure, these functions also depend on V (i.e., type B worker's value at $\underline{p}$), as well as primitive parameters (w, y, r). Whenever there is no confusion, we will typically not make the dependence notationally explicit. However, such dependence will be analyzed in detail in Claim 5 below.

Notice that we have given close-form formula for $\hat{a}, \hat{v}$ and $\hat{s}$ in Lemmas 6 through 8. The next lemma characterizes the firm's value function on $(0, \underline{p})$,[17] as well as the lower cutoff $\underline{p}$ in the firm's strategy.

Lemma 9. *In any equilibrium (a^*, s^*), the corresponding $\check{a}$ and $\check{s}$ defined in (38) do not depend on $\underline{h}$. Moreover, $\check{\pi}$ satisfies that*

$$\check{\pi}(\alpha; \underline{h}) = \frac{\alpha \underline{h}}{1 + \alpha \underline{h}} \check{\pi}_1(\alpha) + \frac{1}{1 + \alpha \underline{h}} \check{\pi}_0(\alpha) \tag{39}$$

where $\check{\pi}_1(\alpha)$ is the unique solution to [18]

$$r \check{\pi}_1(\alpha) = y - \check{a}(\alpha) + \check{\pi}_1'(\alpha) \check{a}(\alpha)^2 \alpha + \frac{1}{2} \check{\pi}_1''(\alpha) \check{a}(\alpha)^2 \alpha^2$$

[17] For all $p > \underline{p}$, indifference/optimality of the firm's behavior implies that its value is 0.

[18] For existence of solution, see Online Appendix.

with boundary conditions $\breve{\pi}_1(0) = \frac{y-1}{r}$ *and* $\breve{\pi}_1(1) = 0$*; and* $\breve{\pi}_0(\alpha)$ *is the unique solution to* [19]

$$r\breve{\pi}_0(\alpha) = y + \frac{1}{2}\breve{\pi}_0''(\alpha)\breve{a}(\alpha)^2\alpha^2$$

with boundary conditions $\breve{\pi}_0(0) = \frac{y}{r}$ *and* $\breve{\pi}_0(1) = 0$.

Finally, $\underline{h}$ *satisfies that*

$$\underline{h} = -\frac{\breve{\pi}_0'(1)}{\breve{\pi}_1'(1)}. \tag{40}$$

Proof. Since $\breve{s}(\alpha; \underline{h}) = 0$ for all α, $\breve{s}$ is independent of $\underline{h}$. Also, it is straightforward to use conditions (27) and (28) in Lemma 8 to verify that $\hat{a}$ is only a function of $\frac{h}{\underline{h}}$ (which is just α), and thus $\breve{a}$ is independent of $\underline{h}$.

Given the law of motion of p_t (from the firm's viewpoint) in (1), by Ito's lemma, we have

$$d\alpha_t = \alpha_t\breve{a}(\alpha_t)\left(\frac{\alpha_t\underline{h}}{1+\alpha_t\underline{h}}\breve{a}(\alpha_t)dt - dB_t^F\right),$$

where dB_t^F a Brownian motion. Then, $\breve{\pi}$ satisfies that

$$\breve{\pi}(\alpha_t; \underline{h}) = \left(y - \frac{\alpha_t\underline{h}}{1+\alpha_t\underline{h}}\breve{a}(\alpha_t)\right)dt + \mathbb{E}_t\left[e^{-rdt}\breve{\pi}(\alpha_{t+dt}; \underline{h})\right].$$

Above, notice that $p_t = \frac{\alpha_t\underline{h}}{1+\alpha_t\underline{h}}$. The HJB equation for $\breve{\pi}$ is

$$r\breve{\pi}(\alpha; \underline{h}) = y - \breve{a}(\alpha)\frac{\alpha\underline{h}}{1+\alpha\underline{h}} + \breve{\pi}'(\alpha; \underline{h})\alpha\breve{a}(\alpha)^2\frac{\alpha\underline{h}}{1+\alpha\underline{h}} + \frac{1}{2}\breve{\pi}''(\alpha; \underline{h})\alpha^2\breve{a}(\alpha)^2, \forall\alpha \in (0,1), \tag{41}$$

with boundary conditions $\breve{\pi}(0; \underline{h}) = \frac{y}{r}$ and $\breve{\pi}(1; \underline{h}) = 0$.

Now we verify that equation (39) is the unique solution to (41).[20] Let us denote the RHS of (39) by $\breve{\pi}_{conj}$. Note that

$$\breve{\pi}_{conj}'(\alpha; \underline{h}) = \frac{\alpha\underline{h}}{1+\alpha\underline{h}}\breve{\pi}_1'(\alpha) + \frac{1}{1+\alpha\underline{h}}\breve{\pi}_0'(\alpha) + \frac{\underline{h}}{(1+\alpha\underline{h})^2}\left[\breve{\pi}_1(\alpha) - \breve{\pi}_0(\alpha)\right],$$

$$\breve{\pi}_{conj}''(\alpha; \underline{h}) = \frac{\alpha\underline{h}}{1+\alpha\underline{h}}\breve{\pi}_1''(\alpha) + \frac{1}{1+\alpha\underline{h}}\breve{\pi}_0''(\alpha) + \frac{2\underline{h}}{(1+\alpha\underline{h})^2}\left[\breve{\pi}_1'(\alpha) - \breve{\pi}_0'(\alpha)\right] - \frac{2\underline{h}^2}{(1+\alpha\underline{h})^3}\left[\breve{\pi}_1(\alpha) - \breve{\pi}_0(\alpha)\right].$$

Direct computation then shows that $\breve{\pi}_{conj}$ satisfies (41). Moreover, it is also easy to verify that $\breve{\pi}_{conj}(0; \underline{h}) = \frac{y}{r}$ and $\breve{\pi}_{conj}(1; \underline{h}) = 0$, so $\breve{\pi}_{conj}$ solves (41) with the associated boundary conditions.

Finally, we prove that in equilibrium $\underline{h} = -\frac{\breve{\pi}_0'(1)}{\breve{\pi}_1'(1)}$. To see this, notice that $\pi^*(p) = 0$ for all $p \geq \underline{p}$, as

[19] For existence of solution, see Online Appendix.
[20] We establish the uniqueness of solution in the Online Appendix.

required by firm optimality. Smooth pasting of π^* at $\underline{p}$ implies that

$$\begin{aligned}
0 &= \check{\pi}'(\alpha;\underline{h})|_{\alpha=1} \\
&= \frac{\underline{h}}{(1+\alpha\underline{h})^2}\left[\check{\pi}_1(\alpha)-\check{\pi}_0(\alpha)\right] + \frac{1}{(1+\alpha\underline{h})^2}\left[\alpha\underline{h}\check{\pi}_1(\alpha)+\check{\pi}_0(\alpha)\right]\Big|_{\alpha=1} \\
&= \frac{1}{(1+\underline{h})^2}\left[\underline{h}\check{\pi}_1(1)+\check{\pi}_0(1)\right].
\end{aligned}$$

Thus, $\underline{h} = -\frac{\check{\pi}_0'(1)}{\check{\pi}_1'(1)}$. □

Before we proceed, we provide a useful result on the linearity of the solution to an ODE w.r.t. its boundary values.

Claim 4. *Let $\mathcal{X} = [0,1]$. Suppose that $y(x)$ defined on $\mathcal{X}$ is the unique solution to the following ODE*

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

with boundary conditions $y(0) = y_0$, $y(1) = y_1$. Moreover, suppose that $g_0(x)$ is the unique solution to

$$a(x)g'' + b(x)g' + c(x)g = 0$$

with boundary conditions $g(0) = 1$, $g(1) = 0$; g_1 is the unique solution to

$$a(x)g'' + b(x)g' + c(x)g = 0$$

with boundary conditions $g(0) = 0$, $g(1) = 1$; g_2 is the unique solution to

$$a(x)g'' + b(x)g' + c(x)g = d(x)$$

with boundary conditions $g(0) = 0$, $g(1) = 0$. Then we must have

$$y(x) = g_0(x)y_0 + g_1(x)y_1 + g_2(x).$$

Proof. The claim can be easily verified by substituting $y(x) = g_0(x)y_0 + g_1(x)y_1 + g_2(x)$ into the first ODE. □

Recall the definition of $\check{a}$ in (38). Lemma 9 indicates that it is independent of $\underline{h}$, and its value only depends on α as well as (V, w, y, r). Define

$$\underline{a}(V) \equiv \check{a}(1; V) \tag{42}$$

That is, if in equilibrium type B worker's value at $\underline{p}$ is V, then $\underline{a}(V)$ is his equilibrium action at belief $\underline{p}$. Note that $\underline{a}$ is independent of the firm's instantaneous payoff y, and that we have dropped its dependence on

w and r for notational convenience.

Also, as in Lemma 9, define

$$\underline{h}(V,y) \equiv -\frac{\check{\pi}_0'(1;V,y)}{\check{\pi}_1'(1;V,y)}. \tag{43}$$

Finally, define

$$\underline{e}(V,y) \equiv y - \underline{a}(V)\frac{\underline{h}(V,y)}{1+\underline{h}(V,y)}. \tag{44}$$

Again, for notational convenience, we have dropped their dependence on w and r.

Claim 5. *The following statements hold.*

1. $0 < \underline{h}(V,y) < \infty$ *whenever* $\underline{e}(V,y) \leq 0$.
2. *For any fixed* $V \in \left[0, \frac{1+w}{r}\right)$, $\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y)$ *is strictly increasing in* y.
3. *For any fixed* $y \in (0,1)$, $\underline{e}(V,y)$ *is strictly decreasing in* V*, satisfying* $\underline{e}(V_m,y) < 0$*; moreover,* $\underline{e}(\cdot,y)$ *always admits a unique zero point.*

Proof. 1. To analyze the sign of $\underline{h}(V,y)$ defined in (40), first we need to understand better $\check{\pi}_0'(1)$ and $\check{\pi}_1'(1)$, which are defined in Lemma 9.

By Claim 4, there exist functions $k_0(\alpha;V)$, $g_0(\alpha;V)$ and $g_1(\alpha;V)$, all independent of y, such that

$$\begin{aligned}\check{\pi}_0(\alpha;V,y) &= \frac{y}{r} - k_0(\alpha;V)\frac{y}{r},\\ \check{\pi}_1(\alpha;V,y) &= \frac{y-1}{r} - g_0(\alpha;V)\frac{y-1}{r} + g_1(\alpha;V),\end{aligned} \tag{45}$$

where $k_0(\alpha;V)$ is the unique solution to

$$rk_0 = \frac{1}{2}k_0''\check{a}(\alpha;V)^2\alpha^2$$

with boundary conditions $k_0(0) = 0$ and $k_0(1) = 1$; $g_0(\alpha;V)$ is the unique solution to

$$rg_0 = \alpha\check{a}(\alpha;V)^2 g_0' + \frac{1}{2}\alpha^2\check{a}(\alpha;V)^2 g_0''$$

with boundary conditions $g_0(0) = 0$ and $g_0(1) = 1$; $g_1(\alpha;V)$ is the unique solution to

$$rg_1 = 1 - \check{a}(\alpha;V) + \alpha\check{a}(\alpha;V)^2 g_1' + \frac{1}{2}\alpha^2\check{a}(\alpha;V)^2 g_1'' \tag{46}$$

with boundary conditions $g_1(0) = g_1(1) = 0$.

The following observations on k_0 and g_0 turn out to be important.

$$k_0'(1;V) > 0, \tag{47}$$

$$g_0'(1;V) > 0, \tag{48}$$

$$k_0'(1;V) - g_0'(1;V) > 0. \tag{49}$$

To prove (47), suppose the contrary. Then there must exist a local maximum point $\alpha_1 \in (0,1]$ of k_0 such that $k_0'(\alpha_1;V) = 0$ and $k_0''(\alpha_1;V) \leq 0$. Since $k_0(\alpha;V) > 0$ for all $\alpha \in (0,1)$,[21] the ODE for $k_0(\alpha;V)$ will be violated at α_1, a contradiction. A similar argument can be used to establish (48).

To prove (49), consider a new function $\delta = k_0(\alpha;V) - g_0(\alpha;V)$. Then δ satisfies the following ODE

$$r\delta = -\alpha\breve{a}(\alpha;V)^2 g_0(\alpha;V) + \frac{1}{2}\alpha^2\breve{a}(\alpha;V)^2\delta''$$

with boundary conditions $\delta(0) = \delta(1) = 0$. A similar argument to the above paragraph shows that $\delta'(1) > 0$, which is equivalent to $k_0'(1;V) - g_0'(1;V) > 0$.

From (44), we know that $\underline{e}(V,y) \leq 0$ implies $\frac{\underline{h}(V,y)}{1+\underline{h}(V,y)} > 0$; so, whenever $\underline{e}(V,y) \leq 0$, we must have $\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y) \leq 0$. Now we argue that $\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y) \leq 0$, in turn, implies that $\underline{h}(V,y) \geq 0$.

Substituting (45) into (43), we get

$$\underline{h}(V,y) = \frac{k_0'(1;V)\frac{y}{r}}{g_0'(1;V)\frac{1-y}{r} + g_1'(1;V)}. \tag{50}$$

Conditions (47) and (48) then imply that

$$\underline{h}(V,y) > 0 \iff \frac{y}{r} \leq \frac{g_1'(1;V) + g_0'(1;V)\frac{1}{r}}{g_0'(1;V)},$$

and $\underline{h}(V,y) = \infty$ when the equality holds. Moreover, conditions (47) and (49) imply that

$$\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y) \leq 0 \iff \frac{y}{r} \leq \frac{k_0'(1;V)\frac{\breve{a}(1;V)}{r} - g_0'(1;V)\frac{1}{r} - g_1'(1;V)}{k_0'(1;V) - g_0'(1;V)}.$$

Consider first the case where $\breve{a}'(1;V) = 0$, so that $\breve{a}(\alpha;V) = 1$ for all $\alpha \in [0,1]$. In this case, the solution to ODE (46) is $g_1(\alpha) \equiv 0$, and condition (50) becomes

$$\underline{h}(V,y) = \frac{k_0'(1;V)\frac{y}{r}}{g_0'(1;V)\frac{1-y}{r}}.$$

Conditions (47), (48) and that $y < 1$ implies that $0 < \underline{h}(V,y) < \infty$.

[21] Suppose that $\min_{\alpha\in(0,1)} k_0(\alpha;V) < 0$, and let $\alpha_1' \in \operatorname{argmin}_{\alpha\in(0,1)} k_0(\alpha;V)$. Then $k_0(\alpha_1') < 0$ while $k_0''(\alpha_1';V) \geq 0$, violating the ODE for k_0, a contradiction.

Consider now the case where $\breve{a}'(1;V) < 0$. Toward showing that $\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y) \leq 0$ implies that $0 < \underline{h}(V,y) < \infty$, it suffices to prove that

$$\frac{k_0'(1;V)\frac{\breve{a}(1;V)}{r} - g_0'(1;V)\frac{1}{r} - g_1'(1;V)}{k_0'(1;V) - g_0'(1;V)} < \frac{g_1'(1;V) + g_0'(1;V)\frac{1}{r}}{g_0'(1;V)}. \tag{51}$$

Rearranging, we know that condition (51) is equivalent to

$$g_1'(1;V) + \frac{1-\breve{a}\,(1;V)}{r}g_0'(1;V) > 0.$$

To show that this condition holds, let us define a function $Y(\alpha) \equiv g_1(\alpha;V) + \frac{1-\breve{a}(1;V)}{r}g_0(\alpha;V)$, and we need to show that $Y'(1) > 0$. It is easy to see that $Y(\alpha)$ satisfies the following ODE

$$rY = 1 - \breve{a}(\alpha;V) + Y'\alpha\breve{a}(\alpha;V)^2 + \frac{1}{2}\alpha^2\breve{a}(\alpha;V)^2Y'' \tag{52}$$

with boundary conditions $Y(0) = 0$ and $Y(1) = \frac{1-\breve{a}(1;V)}{r}$.

Suppose (by contradiction) that $Y'(1) < 0$. Then there must exist $\alpha_1 \in (0,1)$ such that $Y(\alpha_1) > Y(1)$, $Y'(\alpha_1) = 0$ and $Y''(\alpha_1) \leq 0$. This implies that

$$1 - \breve{a}(\alpha_1;V) \geq rY(\alpha_1) > rY(1) = 1 - \breve{a}(1;V).$$

But this is a contradiction to $\breve{a}(\alpha;V)$ being decreasing in α.[22]

Suppose (by contradiction, again) that $Y'(1) = 0$. Since $Y(1) = \frac{1-\breve{a}(1;V)}{r}$, by (52), we must have $Y''(1) = 0$. Differentiating both sides of (52) at $\alpha = 1$ and substituting $Y(1) = \frac{1-\breve{a}(1;V)}{r}$ and $Y'(1) = Y''(1) = 0$ into the result,[23] we can get

$$\breve{a}'(1;V) = \frac{1}{2}\breve{a}^2(1;V)Y'''(1).$$

Because $\breve{a}'(1;V) < 0$ by Lemma 6(1), we have $Y'''(1) < 0$. Together with $Y(1) = Y'(1) = Y''(1) = 0$, we know that there must exist $\alpha_1 \in (0,1)$ such that $Y(\alpha_1) > Y(1)$, $Y'(\alpha_1) = 0$ and $Y''(\alpha_1) \leq 0$. So the same contradiction as in the previous case arises.

2. By equation (50), we have

$$\begin{aligned}\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)}\underline{e}(V,y) &= y\frac{1+\underline{h}(V,y)}{\underline{h}(V,y)} - \breve{a}(1;V)\\ &= \frac{k_0'(1;V) - g_0'(1;V)}{k_0'(1;V)}y + \frac{g_0'(1;V)}{k_0'(1;V)} + r\frac{g_1'(1;V)}{k_0'(1;V)} - \breve{a}(1;V).\end{aligned}$$

[22] Recall the definition of $\breve{a}$ in (38) and that the equilibrium action is decreasing in belief by Lemma 8.
[23] $Y'''(1)$ must exist because $\breve{a}$ in (52) is differentiable at $\alpha = 1$.

Part 2 of the claim follows easily from conditions (47) and (49).

3. [TO BE ADDED]

□

Lemma 10. *In any equilibrium* (a^*, s^*)*,*

- *if* $V = 0$*, then* $\underline{e}(0, y) \leq 0$ *and* $\underline{p} = \bar{p}$*, where* $\underline{p}$ *is given by* (40)*.*
- *if* $V > 0$*, then* $\underline{e}(V, y) = 0$ *and* $\underline{p} < \bar{p}$*, where* $\underline{p}$ *is given by* (40) *and*

$$\bar{p} = 1 - e^{-Vy}(1 - \underline{p}).$$

For any (w, y, r)*, there exists unique* $(\underline{p}, \bar{p}, V)$ *satisfying the above equilibrium conditions.*

Proof. Suppose first that $v^*(\underline{p}) = V = 0$. By Claim 3, v^* is strictly decreasing on $(\underline{p}, \bar{p})$. Since $v^*(p) \geq 0$ for all p, we must have $\underline{p} = \bar{p}$. Further, for $p \in (0, \underline{p})$, Lemma 4 indicates that the firm's value function π^* satisfies

$$r\pi^*(p) = y - a^*(p)p + \frac{1}{2}p^2(1-p)^2 a^*(p)^2 \pi^{*\prime\prime}(p).$$

As $p \to \underline{p}$, we have $\pi^{*\prime\prime}(p) \geq 0$ (i.e., the firm's value function must be locally convex at the boundary, for otherwise $\pi^*(p) < 0$ for p close enough to $\underline{p}$), which implies that $y - a^*(p)p - r\pi^*(p) \leq 0$. Since $a^*(p)$ is continuous on $(0, \underline{p}]$ and $\pi^*(\underline{p}) = 0$,[24] we have $\underline{e}(0, y) = y - a^*(\underline{p})\underline{p} \leq 0$.

Suppose now that $V > 0$. Using equation (13), as well as the conditions that $v^*(\bar{p}) = 0$ and $v^*(\underline{p}) = V$, one can find that $\bar{p} = 1 - e^{-Vy}(1 - \underline{p})$ (note that $\bar{p} > \underline{p}$). By Claim 2, we know that $y - a^*(p)p = 0$ for all $p \in (\underline{p}, \bar{p})$; by Lemma 6, a^* is continuous at $\underline{p}$, so that $\underline{e}(V, y) = y - a^*(\underline{p})\underline{p} = 0$.

Fix any (w, y, r). If $\underline{e}(0, y) \leq 0$, then Claim 5(1) tells us that $\underline{p}$ determined by (40) is well-defined. So setting $V = 0$ and $\bar{p} = \underline{p}$ satisfies the first equilibrium condition. Moreover, Claim 5(2) implies that it is impossible to have any $V > 0$ satisfying $\underline{e}(V, y) = 0$. On the other hand, if $\underline{e}(0, y) > 0$, by Claim 5(2), there exists a unique $V \in (0, V_m)$ such that $\underline{e}(V, y) = 0$. The previous calculation implies that such a V, together with $\underline{p}$ determined by (40) and $\bar{p} = 1 - e^{-Vy}(1 - \underline{p})$, uniquely satisfies the second equilibrium condition. □

A.4 Proofs of the Main Results

Proof of Theorem 1. Fix any (w, y, r), and let $(\underline{p}, \bar{p}, V)$ be the unique vector that satisfies the equilibrium condition in Lemma 10. We will show that the strategy profile, parameterized by $(\underline{p}, \bar{p}, V)$, indeed constitutes an equilibrium. We consider the cases $V = 0$ and $V > 0$ separately.

[24] Lemma 6 implies that a^* is continuous on $(0, \bar{p})$; in the current case where $\bar{p} = \underline{p}$, a^* is continuous on $(0, \underline{p})$. Because we treat two strategies that are almost surely equal as the same, it is without loss to assume that a^* is continuous on $(0, \underline{p}]$.

Case 1. $V = 0$ and $\underline{e}(0, y) \leq 0$.

Consider the following strategies. Let $a^*(p)|_{(0,\underline{p})}$ be given by Lemma 8, $a^*(p)|_{(\underline{p},1)} = 1$; $s^*(p)|_{(0,\underline{p})} = 0$, $s^*(p)|_{[\underline{p},1)} = \infty$. Note that these strategies satisfy the properties claimed in Theorem 1. We want to show that (a^*, s^*) is an equilibrium. First, given the firm's strategy s^*, since a^* (and the induced v^*) are solved from Lemma 8 and thus satisfy the HJB in Lemma 1, we know that a^* satisfies worker optimality.

To see that s^* is optimal for the firm, let us define $\pi(p)$ in the following way: on $(0, \underline{p})$, $\pi(p)$ solves the following ODE

$$r\pi = y - pa^* + \frac{1}{2}\pi'' p^2(1-p)^2 a^{*2} \tag{53}$$

with boundary conditions $\pi(0) = \frac{y}{r}$ and $\pi(\underline{p}) = 0$; moreover, $\pi(p)|_{(\underline{p},1)} = 0$. The construction of $\underline{h}$ (and thus $\underline{p}$) in Lemma 9 ensures that $\pi'(\underline{p}) = 0$. Lemma 3 in Kuvalekar and Lipnowski (2018) provides sufficient condition for s^* to be the firm's best response to a^*. In particular, we need to check the following conditions:

1) $\pi \geq 0$;

2) π is C^1 and piecewise C^2;

3) $\pi|_{(\underline{p},1)} = 0$;

4) For a.e. $p \in (0, 1)$, $y - pa^* \leq r\pi - \frac{1}{2}\pi'' p^2(1-p)^2 a^{*2}$ with equality if $p \leq \underline{p}$.

It is easy to verify that conditions 2), 3) and 4) are satisfied. Now we prove 1).

Recall that $e(0, y) = y - a^*(\underline{p})\underline{p} \leq 0$. Suppose first that $y - a^*(\underline{p})\underline{p} < 0$. We know from the ODE for π that $\pi''(\underline{p}) > 0$, so that $\pi(p) > 0$ for p close to $\underline{p}$. Suppose (by contradiction) that there exists p_0 such that $\pi(p_0) < 0$. Then there must exist a local minimum point p_1 and a local maximum point p_2 of π with $0 < p_1 < p_2 < \underline{p}$, such that $\pi(p_1) < 0$ and $\pi(p_2) > 0$. Notice that

$$a^*(p_1)p_1 \geq y - r\pi(p_1) > y - r\pi(p_2) \geq a^*(p_2)p_2$$

where the first (third) inequality follows from $\pi''(p_1) \geq 0$ ($\pi''(p_2) \leq 0$), and the second inequality follows from $\pi(p_1) < 0 < \pi(p_2)$. Then, by a similar argument to that of Lemma 6(1), one can show that $a^*(p)p$ must be decreasing on $[p_2, \underline{p}]$. However, since $y - a^*(\underline{p})\underline{p} < 0$, we have

$$a^*(\underline{p})\underline{p} > y \geq y - r\pi(p_2) \geq a^*(p_2)p_2,$$

a contradiction. Thus, when $y - a^*(\underline{p})\underline{p} < 0$, we must have $\pi(p) \geq 0$ for all p.

Suppose now that $y - a^*(\underline{p})\underline{p} = 0$. Lemma 8 and Claim 5(3) imply that $a^*(\underline{p}) < 1$. The argument in the last part of Proposition 2's proof (see below) shows that $\frac{d}{dp}\left[y - a^*(p)p\right]\Big|_{p=\underline{p}} > 0$. Taking derivatives on both sides on (53), we can see that $\pi'''(\underline{p}) < 0$. Together with $\pi(\underline{p}) = \pi'(\underline{p}) = \pi''(\underline{p}) = 0$, we know that $\pi(p) > 0$ for p less than but close to $\underline{p}$. Then the same argument as above shows that $\pi(p) \geq 0$ for all p.

Case 2. $V > 0$ and $\underline{e}(V, y) = 0$.

Consider the following strategies. Let $a^*(p)|_{(0,\underline{p})}$ be given by (27) in Lemma 8, $a^*(p)|_{(\underline{p},\bar{p})} = \frac{y}{p}$ as in (12), and $a^*(p)|_{(\bar{p},1)} = 1$; also, let $s^*(p)|_{(0,\underline{p})} = 0$, $s^*(p)|_{(\underline{p},\bar{p})}$ be given by (14),[25] and $s^*(p)|_{[\bar{p},1)} = \infty$. Note that these strategies satisfy the properties claimed in Theorem 1. An argument analogous to Case 1 verifies worker and firm optimality.

Finally, equilibrium uniqueness follows from the uniqueness part of Lemma 10. □

Proof of Proposition 1. As the proof of Lemma 10 shows, $\underline{e}(V, y; w, r)$ depends on (V, w, r) only through $V - \frac{w}{r}$. Let $u_0(y)$ be such that whenever $V - \frac{w}{r} = u_0(y)$, $\underline{e}(V, y; w, r) = 0$. Note that $u_0(y)$ is well-defined and unique because of Claim 5(3). Now define

$$\bar{w}(y; r) = \max\{0, -ru_0(y)\}.$$

Notice that whenever $w > \bar{w}(y; r)$, $0 - \frac{w}{r} < u_0(y)$, so that $\underline{e}(0, y; w, r) > 0$. Then, by Lemma 10 and Theorem 1, $\underline{p} < \bar{p}$ in the equilibrium whenever $w > \bar{w}(y, r)$. That $\bar{w}(y, r)$ is decreasing in y follows directly from Claim 5(1) and Claim 5(2). □

Proof of Proposition 2. Fix (w, y, r) and let (a^*, s^*) be the unique equilibrium with the associated $(\underline{p}, \bar{p}, V)$. By Lemma 8, there exists $p_1 \leq \underline{p}$ s.t.

$$a^*(p) \begin{cases} = 1, & \text{if } p \in (0, p_1) \\ < 1, & \text{if } p \in (p_1, \underline{p}) \end{cases}.$$

Obviously $y - a^*(p)p$ is decreasing on $(0, p_1)$. Moreover, by Lemma 6(1), there exists $p_2 \in [p_1, \underline{p}]$ s.t. $y - a^*(p)p$ is decreasing on (p_1, p_2) and increasing on $(p_2, \underline{p})$. Finally, by Lemma 6(2), $y - a^*(p)p$ is constant on $(\underline{p}, \bar{p})$.

By the continuity of a^* on $(0, \bar{p})$, we conclude that there exists p_2, s.t. $y - a^*(p)p$ is decreasing on $(0, p_2)$ and increasing on $(p_2, \bar{p})$.

In fact, whenever (w, y, r) is such that $a^*(\underline{p}) < 1$, we have that $y - a^*(p)p$ is *strictly* increasing at any p less than but close to $\underline{p}$. To see this, note that derivations similar to that of (22) gives us

$$\operatorname{sgn}\left[\frac{d}{dp}\left(y - a^*(p)p\right)\right] = \operatorname{sgn}\left[\frac{d}{dh}\left(-\hat{v}'(h)(1+h)\right)\right] = \operatorname{sgn}\left[1 - \frac{2(r\hat{v}(h) - w)}{\hat{a}(h)p(h)}\right]. \tag{54}$$

If $\hat{v}(\underline{h}) = V = 0$ (i.e., the equilibrium does not involve stochastic termination), it is obvious from (54) that $y - a^*(p)p$ is strictly increasing in a left neighborhood of $\underline{p}$. On the other hand, if $\hat{v}(\underline{h}) = V > 0$, let us suppose (by contradiction) that (54) takes a non-positive value at $\underline{p}$; for brevity, suppose that it is strictly negative.[26] Then by Lemma 6(1), $y - a^*(p)p$ is decreasing and thus positive on $(0, \underline{p})$. Recall the

[25] The constant C in (13) is found by solving $v^*(\underline{p}) = V$. For $s^*(p)$ to be well-defined, (14) also requires that $w + \frac{y}{2} \geq rV$. This is established later on in the proof of Proposition 2; see footnote 27.

[26] The case where the derivative is 0 can be dealt with using an approach similar to this section, by taking higher-order derivatives of π

defining ODE of π in (53). The positivity of $y - a^*(p)p$ implies the positivity of π on $(0, \underline{p})$, because $\pi(p) = \mathbb{E}_\tau \left[\int_0^\tau (y - a^*(p_t)p_t)e^{-rt}dt | p_0 = p\right]$ where τ is first time p_t reaches $\underline{p}$. From (53), we know that $\pi''(\underline{p}) = 0$ because $\pi'(\underline{p}) = y - a^*(\underline{p})\underline{p} = 0$. Taking derivative on both sides of (53) at $p = \underline{p}$, we have

$$r\pi'(\underline{p}) = \frac{d}{dp}\left[y - a^*(p)p\right]\bigg|_{p=\underline{p}} + \frac{1}{2}\pi'''\underline{p}^2(1-\underline{p})^2 a^*(\underline{p})^2.$$

This implies that $\pi'''(\underline{p}) > 0$. Together with $\pi(\underline{p}) = \pi'(\underline{p}) = \pi''(\underline{p}) = 0$, we have that for p less than but close enough to $\underline{p}$, we have $\pi(p) < 0$. But this is a contradiction to the positivity of π we just deduced. So we conclude that $y - a^*(p)p$ is strictly increasing in a left neighborhood of $\underline{p}$.[27] $\square$

Proof of Proposition 3. Let us first derive some properties of any equilibrium that involves replacement. Within the relationship with a particular worker, at each instant the firm is comparing his payoff from staying in the current relationship, $\pi^*(p)$, from taking the outside option, 0, and from replacing the worker, $\pi_R = \pi^*(p_0) - c$. If an equilibrium involves replacement, it must be that $\pi_R \geq 0$. Taking the equilibrium value of π_R as given, the players are playing the game in our baseline model specified in Section 2 with the firm's outside value being π_R; on the other hand, any π_R induces a (static) Markov equilibrium in a given relationship. It is easy to see that our baseline model (without replacement) with firm instantaneous payoff y and outside value π_R is strategically equivalent to one with $y_R \equiv y - r\pi_R$ and outside value 0.[28] We find a stationary Markov equilibrium with replacement if the (static) equilibrium (a^*, s^*) induced by π_R satisfies

$$\pi_R = \pi^*(\underline{p}) = \pi^*(p_0) - c. \tag{55}$$

where $\underline{p}$ is the lowest belief above which $s^*(p) > 0$, as in Theorem 1. The aforementioned strategic equivalence also implies that, instead of searching over π_R to check if (55) holds, we can fix the outside value to 0 while varying y_R, until the induced (static) equilibrium satisfies

$$\pi^*(p_0) = c. \tag{56}$$

Now we go back to our baseline model and examine when condition (56) can be satisfied. Given our previous analysis, one can show that the firm's value function, $\pi^*(\cdot; y_R)$, in the unique Markov equilibrium delivered by Theorem 1 is continuous in y_R. Moreover, for any fixed p_0, $\pi^*(p_0; y_R) \to 0$ as $y_R \to 0$. Define

$$\bar{c} = \max_{y_R \in [0,y]} \pi^*(\cdot; y_R).$$

- If $c > \bar{c}$, there does not exist any $y_R \in [0, y]$ s.t. (56) holds; in other words, there does not exist any $\pi_R \geq 0$ s.t. (55) holds. As a result, no stationary Markov equilibrium involves replacement because such an option is too costly. The unique stationary Markov equilibrium is characterized by Theorem 1, in which the firm takes its outside option after firing the first worker.

[27] From (54), an immediate implication of this result is that $w + \frac{y}{2} > rV$. This corresponds to footnote 25.
[28] Namely, any Markov equilibrium in one game is also a Markov equilibrium in the other.

- If $c \leq \bar{c}$, intermediate value theorem delivers some $y_R^* \in [0, y]$ s.t. (56) holds. Our previous argument implies that the static equilibrium in our baseline model under parameter values (w, y_R^*, r) constitutes a stationary Markov equilibrium, in which the firm replaces the worker if his reputation is bad enough, but never takes its outside option.

□

Proof of Proposition 4. In the Online Appendix, we provide explicit bounds on $w'(p)$ and p_m, under which the worker's value function v^* in any Markov equilibrium (a^*, s^*) satisfies

$$1 + v^{*\prime}(p)\, p\,(1-p) \geq 0, \forall p \leq p_m.$$

Lemma 2 and condition (15) then imply that $a^*(p) = 1$ for all $p \leq p_m$. Intuitively, if the incentive provided by wage variation (capture by w') is not strong enough, and if the worker's reputation is good enough (so that job is sufficiently secure), then type B worker wants to shirk to the maximum level.

The same argument as that in the proof of Lemma 7 can then establish that the firm's strategy s^* in any Markov equilibrium fits Theorem 1's description.[29]

With respect to the worker's strategy a^*, we mentioned above that $a^*(p) = 1$ for all $p \leq p_m$, whereas for all $p > p_m$, $w(p)$ is by assumption constant, so that the analyses in Lemmas 6 and 8 apply. Consequently, the worker's strategy a^* in any Markov equilibrium fits Theorem 1's description.

Finally, since the equilibrium strategies have the same structure as that Theorem 1, almost the same analysis as the proof of Proposition 2 establishes that the average productivity $y - a^*(p)p$ is a U-shaped function of p in any Markov equilibrium.

□

[29] The upper bound we provide for p_m is less than y. Thus, for all $p \leq p_m$ where wage could vary, we always have $s^*(p) = 0$ because at those beliefs the firm's instantaneous payoff $y - ap$ is always positive regardless of the worker's action. The analysis for $p > p_m$ where wage is constant follows closely with the proof of Lemma 7.